DOCUMENT RESUME

ED 064 146

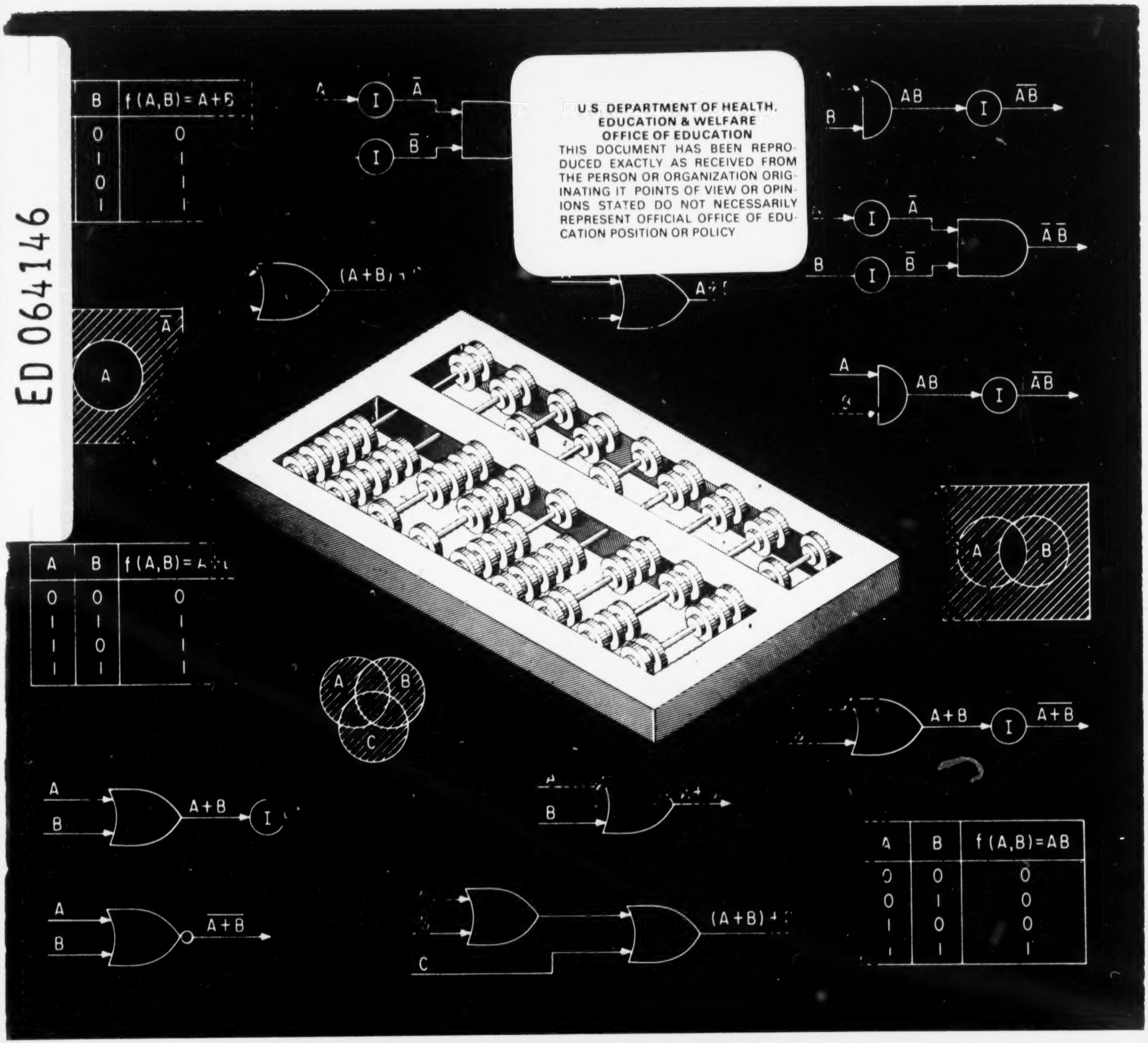
SE 014 117

TITLE	Mathematics, Vol. 3.
INSTITUTION	Department of the Navy, Washington, D.C. Bureau of Naval Personnel.
REPORT NO	NavPers-10073-A
PUB DATE	69
NOTE	185p.
EDRS PRICE	MF-$0.65 HC-$6.58
DESCRIPTORS	*College Mathematics; *Computer Oriented Programs; Computer Science; *Instructional Materials; *Mathematics; Number Systems; Set Theory; Statistics
IDENTIFIERS	Boolean Algebra; Matrices

ABSTRACT

The third of three volumes of a mathematics training course for Navy personnel, this text emphasizes topics needed in understanding digital computers and computer programing. The text begins with sequences and series, induction and the binomial theorem, and continues with two chapters on statistics. Arithmetic operations in number systems other than the decimal system are covered. Set theory and Venn diagrams lead into Boolean algebra; the text concludes with a chapter on matrices and determinants. Related documents are SE 014 115 and SE 014 116. (JM)

ED 064146



U.S. DEPARTMENT OF HEALTH,
EDUCATION & WELFARE
OFFICE OF EDUCATION
THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL OFFICE OF EDUCATION POSITION OR POLICY

MATHEMATICS, VOL. 3

BUREAU OF NAVAL PERSONNEL

RATE TRAINING MANUAL NAVPERS 10073-A

SE 014 117

PREFACE

The purpose of this training manual is to aid those personnel who need an extension of the knowledge of mathematics gained from Mathematics, Vol. 1, NavPers 10069-C, and Mathematics, Vol. 2, NavPers 10071-B. Although the text is not directed exclusively toward any one specific specialty, an attempt has been made to emphasize those topics most likely to be needed as background material in understanding digital computers and computer programing.

This training manual was prepared by the Navy Training Publications Center, Millington, Tennessee, for the Bureau of Naval Personnel.

1969 Edition

Stock Ordering No.
0500-025-2010

For sale by the Superintendent of Documents, U.S. Government Printing Office
Washington, D.C., 20402 - Price $1.75

THE UNITED STATES NAVY

GUARDIAN OF OUR COUNTRY

The United States Navy is responsible for maintaining control of the sea and is a ready force on watch at home and overseas, capable of strong action to preserve the peace or of instant offensive action to win in war.

It is upon the maintenance of this control that our country's glorious future depends; the United States Navy exists to make it so.

WE SERVE WITH HONOR

Tradition, valor, and victory are the Navy's heritage from the past. To these may be added dedication, discipline, and vigilance as the watchwords of the present and the future.

At home or on distant stations we serve with pride, confident in the respect of our country, our shipmates, and our families.

Our responsibilities sober us; our adversities strengthen us.

Service to God and Country is our special privilege. We serve with honor.

THE FUTURE OF THE NAVY

The Navy will always employ new weapons, new techniques, and greater power to protect and defend the United States on the sea, under the sea, and in the air.

Now and in the future, control of the sea gives the United States her greatest advantage for the maintenance of peace and for victory in war.

Mobility, surprise, dispersal, and offensive power are the keynotes of the new Navy. The roots of the Navy lie in a strong belief in the future, in continued dedication to our tasks, and in reflection on our heritage from the past.

Never have our opportunities and our responsibilities been greater.

CONTENTS

Chapter Page

1. Sequence and series 1
2. Mathematical induction and the binomial theorem 20
3. Descriptive statistics 29
4. Statistical inference 48
5. Number systems 71
6. Sets and subsets 108
7. Boolean algebra 122
8. Boolean simplification 142
9. Matrices and determinants 162

Index 178

ACTIVE DUTY ADVANCEMENT REQUIREMENTS

REQUIREMENTS *	E1 to E2	E2 to E3	#† E3 to E4	# E4 to E5	† E5 to E6	† E6 to E7	† E7 to E8	† E8 to E9
SERVICE	4 mos. service—or completion of recruit training.	6 mos. as E-2.	6 mos. as E-3.	12 mos. as E-4.	24 mos. as E-5.	36 mos. as E-6. 8 years total enlisted service.	36 mos. as E-7. 8 of 11 years total service must be enlisted.	24 mos. as E-8. 10 of 13 years total service must be enlisted.
SCHOOL	Recruit Training.		Class A for PR3, DT3, PT3, AME 3, HM 3			Class B for AGC MUC, MNC.		
PRACTICAL FACTORS	Locally prepared check-offs.	Records of Practical Factors, NavPers 1414/1, must be completed for E-3 and all PO advancements.						
PERFORMANCE TEST			Specified ratings must complete applicable performance tests before taking examinations.					
ENLISTED PERFORMANCE EVALUATION	As used by CO when approving advancement.		Counts toward performance factor credit in advancement multiple.					
EXAMINATIONS**	Locally prepared tests.	See below.	Navy-wide examinations required for all PO advancements.				Navy-wide, selection board.	
NAVY TRAINING COURSE (INCLUDING MILITARY REQUIREMENTS)		Required for E-3 and all PO advancements unless waived because of school completion, but need not be repeated if identical course has already been completed. See NavPers 10052 (current edition).					Correspondence courses and recommended reading. See NavPers 10052 (current edition).	
AUTHORIZATION	Commanding Officer		U.S. Naval Examining Center			Bureau of Naval Personnel		

* All advancements require commanding officer's recommendation.

† 1 year obligated service required for E-5 and E-6; 2 years for E-6, E-7, E-8 and E-9.

Military leadership exam required for E-4 and E-5.

** For E-2 to E-3, NAVEXAMCEN exams or locally prepared tests may be used.

INACTIVE DUTY ADVANCEMENT REQUIREMENTS

REQUIREMENTS *	E1 to E2	E2 to E3	E3 to E4	E4 to E5	E5 to E6	E6 to E7	E8	E9
TOTAL TIME IN GRADE	4 mos.	6 mos.	15 mos.	18 mos.	24 mos.	36 mos.	36 mos.	24 mos.
TOTAL TRAINING DUTY IN GRADE †	14 days	14 days	14 days	14 days	28 days	42 days	42 days	28 days
PERFORMANCE TESTS			Specified ratings must complete applicable performance tests before taking examination.					
DRILL PARTICIPATION	Satisfactory participation as a member of a drill unit.							
PRACTICAL FACTORS (INCLUDING MILITARY REQUIREMENTS)	Record of Practical Factors, NavPers 1414/1, must be completed for all advancements.							
NAVY TRAINING COURSE (INCLUDING MILITARY REQUIREMENTS)	Completion of applicable course or courses must be entered in service record.							
EXAMINATION	Standard Exam	Standard Exam or Rating Training.	Standard Exam required for all PO Advancements.			Standard Exam, Selection Board. Also pass Mil. Leadership Exam for E-4 and E-5.		
AUTHORIZATION	Commanding Officer		U.S. Naval Examining Center			Bureau of Naval Personnel		

* Recommendation by commanding officer required for all advancements.

† Active duty periods may be substituted for training duty.

CHAPTER 1

SEQUENCE AND SERIES

A collection or set of numbers, arranged in order, according to some pattern or law, so that one number can be identified as the first and another as the second, and so forth, is referred to as a sequence. The word sequence is sometimes replaced by the word progression, but we will use sequence. The set of natural numbers forms a sequence; that is, 1, 2, 3, ... is a sequence. Each number in the sequence is called a term, and we will represent the first term of a sequence by the letter (a) and the last term by the letter (ℓ).

ARITHMETIC SEQUENCES

An arithmetic sequence is a sequence in which each term may be determined from the preceding term by the addition of a constant. This constant, called the common difference, is designated by the letter (d) and will maintain the same value throughout the sequence.

An arithmetic sequence, then, may be indicated by $a, a + d, a + 2d, a+3d, \ldots, a+(n-1)d$, where there are n terms in the sequence. In the sequence

$$-1, 3, 7, \ldots$$

the common difference (d) is 4. The first term (a) is -1; therefore, the second term is

$$a + d = -1 + 4 = 3$$

The third term is

$$a + 2d = -1 + 2(4) = 7$$

and the fourth term is

$$a + 3d = -1 + 3(4) = 11$$

EXAMPLE: Find the next three terms in the sequence 5, 9, 13, ...

SOLUTION: The first term (a) is 5 and the difference (d) is 4; therefore, write

$$a = 5$$

$$d = 4$$

then, the fourth term is

$$\begin{aligned} a + (n - 1)d &= 5 + (4 - 1)\,4 \\ &= 17 \end{aligned}$$

the fifth term is

$$\begin{aligned} a + (n - 1)d &= 5 + (5 - 1)\,4 \\ &= 21 \end{aligned}$$

and the sixth term is

$$5 + (6 - 1)\,4 = 25$$

We are often interested in finding a specific term of a sequence. We usually refer to this term as the n^{th} term. In cases where the n^{th} term is the last term of a sequence, we write

$$\ell = a + (n - 1)d$$

In this formula there are four unknowns. If we know any three of them, we may find the fourth.

EXAMPLE: Find the 20^{th} term of the sequence

$$1, 3, 5, 7, \ldots .$$

SOLUTION: We know that

$$a = 1$$

$$d = 2$$

$$n = 20$$

Therefore,

$$\begin{aligned} \ell &= a + (n - 1)d \\ &= 1 + (20 - 1)2 \\ &= 39 \end{aligned}$$

Notice that we let the n^{th} (20^{th}) term be the last term.

EXAMPLE: Find the number of terms in a sequence if the last term is 37, the difference is 5, and the first term is -13.

SOLUTION: We know that

$$a = -13$$

$$d = 5$$

$$\ell = 37$$

We solve

$$\ell = a + (n - 1)d$$

for

$$n$$

by writing

$$\ell = a + (n - 1)d$$

$$\ell - a = (n - 1)d$$

$$\frac{\ell - a}{d} = n - 1$$

$$\frac{\ell - a}{d} + 1 = n$$

By substitution

$$\begin{aligned} n &= \frac{\ell - a}{d} + 1 \\ &= \frac{37 - (-13)}{5} + 1 \\ &= \frac{50}{5} + 1 \\ &= 10 + 1 \\ &= 11 \end{aligned}$$

There are cases where we desire the first term of a sequence when only two terms are known.

EXAMPLE: What is the first term of the sequence if the third term is 8 and the sixth term is 20?

SOLUTION: We write

$$\_\ \_\ \underline{8}\ \_\ \_\ \underline{20}$$

and consider a sequence where

$$a = 8$$

$$\ell = 20$$

and

$$n = 4$$

Therefore,

$$20 = 8 + (4 - 1)d$$

then

$$\begin{aligned} d &= \frac{20 - 8}{3} \\ &= 4 \end{aligned}$$

Now consider a sequence where

$$\ell = 20$$

$$d = 4$$

and

$$n = 6$$

then write

$$20 = a + (6 - 1)4$$

$$20 - 20 = a$$

$$a = 0$$

PROBLEMS: Write the next two terms in the following sequences.

1. 18, 21, 24, . . .
2. -19, -16, -13, . . .
3. $x, x + 2, x + 4, \ldots$
4. $\sqrt{2} + 3, \sqrt{2} + 7, \sqrt{2} + 11, \ldots$

Find the term asked for in the following sequences.

5. Seventh term of $-\frac{1}{2}, 0, \frac{1}{2}, \ldots$

6. Twenty-fifth term of -19, -10, -1, . . .

7. Fifth term of $-6x^2, -2x^2, 2x^2, \ldots$

8. Which term of -2, 3, 8, . . . is 88?

ANSWERS:

1. 27, 30

2. -10, -7

3. x + 6, x + 8

4. $\sqrt{2} + 15, \sqrt{2} + 19$

5. $\frac{5}{2}$

6. 197

7. $10x^2$

8. 19th

ARITHMETIC MEANS

In the sequence 1, 3, 5, 7, the terms 3 and 5 occur between the first term 1 and the last term 7 and are designated the means. Generally, the terms which occur between two given terms are called the means.

If we are given the first term (a) and the last term (ℓ) in a sequence of n terms, then there are (n - 2) means between a and ℓ. There can be any number of means between two given terms of a sequence, depending on the difference between adjacent terms.

To determine the means of a sequence we use the formula

$$\ell = a + (n - 1)d$$

EXAMPLE: Insert two arithmetic means between 6 and 12.

SOLUTION: We know

$$a = 6$$

$$\ell = 12$$

and that there are two means in this sequence; therefore,

$$n = 4$$

because the means plus two (the first and last terms) is the number of terms in the sequence. We now determine the difference by writing

$$\ell = a + (n - 1)d$$

$$\ell - a = (n - 1)d$$

$$\frac{\ell - a}{n - 1} = d$$

and by substitution

$$d = \frac{12 - 6}{4 - 1}$$

$$d = \frac{6}{3}$$

$$d = 2$$

We now add this difference to the first term to obtain the second term and the difference to the second term to obtain the third term as follows:

$$6 + 2 = 8 = \text{second term}$$

$$8 + 2 = 10 = \text{third term}$$

and find the means are 8 and 10.

We could also have used the general form of a sequence (a, a + d, a + 2d, a + 3d, . . .) and added the difference to the first term to obtain the second term and added two times the difference to the first term to obtain the third term.

If we use the same first and last terms, that is,

$$a = 6$$

and

$$\ell = 12$$

but now ask for six means, we still use the same formula

$$\ell = a + (n - 1)d$$

and in this case the number of terms is the six means plus the first and last term or

$$n = 8$$

Therefore,

$$d = \frac{\ell - a}{n - 1}$$

$$= \frac{12 - 6}{8 - 1}$$

$$= \frac{6}{7}$$

and the means are

$$a + d = 6 + \frac{6}{7}$$

$$a + 2d = 6 + \frac{12}{7}$$

$$a + 3d = 6 + \frac{18}{7}$$

$$a + 4d = 6 + \frac{24}{7}$$

$$a + 5d = 6 + \frac{30}{7}$$

$$a + 6d = 6 + \frac{36}{7}$$

The previous examples demonstrate that there can be any number of means between two given terms of a sequence, depending on the number of terms and the difference.

PROBLEMS: Insert the indicated number of means in the following:

1. Three, between 3 and 19

2. Two, between -10 and -4

3. Five, between -2 and 2

4. Two, between the first and fourth terms if the fourth term is six and the seventh term is eleven.

5. A secretary can type 3 words per minute faster for each half-hour she types. If she starts at 8:30 a.m. at the rate of 35 words per minute, how fast is she typing at 10:00 a.m.?

ANSWERS:

1. 7, 11, 15

2. -8, -6

3. $-\frac{4}{3}, -\frac{2}{3}, 0, \frac{2}{3}, \frac{4}{3}$

4. $\frac{8}{3}, \frac{13}{3}$

5. 44 words per minute

ARITHMETIC SERIES

When we add all the terms of a sequence, we call this indicated sum a series. We will use the symbol S_n to designate the indicated sum of n terms of a sequence. To derive a formula for S_n we may write the terms of a series as

$$S_n = a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d]$$

Notice that when we had the second term we added the difference to obtain the third term. If we had the third term and desired the second term we would have to subtract the difference. Therefore, if we write the series with the last term first, we subtract the difference for each succeeding term. Then, using ℓ to represent the last term, write

$$S_n = \ell + (\ell - d) + (\ell - 2d) + \cdots + [\ell - (n - 1)d]$$

Now add the two equations, term by term, and find

$$S_n = a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d]$$

$$S_n = \ell + (\ell - d) + (\ell - 2d) + \cdots + [\ell - (n - 1)d]$$

and

$$2S_n = (a + \ell) + (a + \ell) + (a + \ell) + \cdots + (a + \ell)$$

where there are n times $(a + \ell)$. Therefore,

$$2S_n = n(a + \ell)$$

$$S_n = \frac{n}{2}(a + \ell)$$

Another way of obtaining the formula for S_n is to write the terms of a series as follows:

$$S_n = a + (a + d) + (a + 2d) + \cdots$$
$$+ (\ell - 2d) + (\ell - d) + \ell$$

then reverse the order of the series and combine both equations as follows:

$$S_n = a + (a + d) + (a + 2d) + \cdots + (\ell - 2d) + (\ell - d) + \ell$$

$$S_n = \ell + (\ell - d) + (\ell - 2d) + \cdots + (a + 2d) + (a + d) + a$$

and find that

$$2S_n = (a + \ell) + (a + \ell) + \cdots + (a + \ell) + (a + \ell) + (a + \ell)$$

$(a + \ell)$ occurs n times which yields

$$2S_n = n(a + \ell)$$

$$S_n = \frac{n}{2}(a + \ell)$$

EXAMPLE: Find the sum of the first 5 terms of the series 2, 6, 10, ...

SOLUTION: We know that

$$a = 2$$

$$d = 4$$

and

$$n = 5$$

and write

$$S_n = \frac{n}{2}(a + \ell)$$

Here, we must determine ℓ by

$$\begin{aligned} \ell &= a + (n - 1)d \\ &= 2 + (5 - 1)\,4 \\ &= 2 + 16 \\ &= 18 \end{aligned}$$

Then,

$$\begin{aligned} S_n &= \frac{n}{2}(a + \ell) \\ &= \frac{5}{2}(2 + 18) \end{aligned}$$

$$\begin{aligned} S_n &= \frac{5}{2}(20) \\ &= 50 \end{aligned}$$

This may be verified by writing

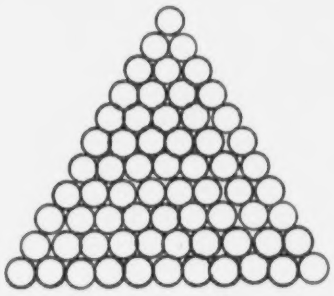
2, 6, 10, 14, 18

and adding the terms to find

$$S_n = 50$$

EXAMPLE: There are eleven pieces of pipe in the bottom row of a stack of pipe which form a triangle. How many pieces of pipe are in the stack?

SOLUTION: We know pipe is stacked as shown.



We have in our stack eleven pieces of pipe on the bottom row and each row up contains one less piece of pipe. There is one piece of pipe on the top.

Therefore, we write

$$a = 1$$

$$n = 11$$

$$d = 1$$

$$\ell = 11$$

Then

$$\begin{aligned} S_n &= \frac{n}{2}(a + \ell) \\ &= \frac{11}{2}(1 + 11) \\ &= \frac{11}{2}(12) \\ &= 66 \text{ pieces of pipe in the stack} \end{aligned}$$

EXAMPLE: Find the sum of 23 terms of the series -3, 2, 7, ...

SOLUTION: We know that

$$a = -3$$

$$d = 5$$

and

$$n = 23$$

Write

$$S_n = \frac{n}{2}(a + \ell)$$

We do not know ℓ, but we do know that

$$\ell = a + (n - 1)d$$

and by substituting for ℓ in the equation for the sum, we have

$$\begin{aligned} S_n &= \frac{n}{2}(a + \ell) \\ &= \frac{n}{2}[a + a + (n - 1)d] \\ &= \frac{n}{2}[2a + (n - 1)d] \end{aligned}$$

Then, using the values we know

$$\begin{aligned} S_n &= \frac{23}{2}[2(-3) + (23 - 1)5] \\ &= \frac{23}{2}[-6 + (22)5] \\ &= \frac{23}{2}[-6 + 110] \\ &= \frac{23}{2}[104] \\ &= 23[52] \\ &= 1{,}196 \end{aligned}$$

In some cases we may have to work from the sum of a sequence in order to determine the sequence.

EXAMPLE: Find the first 4 terms of the sequence if

$$a = 2$$

$$\ell = 18$$

$$S_n = 200$$

SOLUTION: We must find d. We write

$$\ell = a + (n - 1)d$$

but notice that there are two unknowns; that is, n and d. We then write

$$S_n = \frac{n}{2}(a + \ell)$$

and by substitution

$$\begin{aligned} 200 &= \frac{n}{2}(2 + 18) \\ 400 &= n(20) \\ 20 &= n \end{aligned}$$

We again write

$$\ell = a + (n - 1)d$$

and substitute, then write

$$\begin{aligned} 18 &= 2 + (20 - 1)d \\ \frac{18 - 2}{19} &= d \\ d &= \frac{16}{19} \end{aligned}$$

We know that

$$a = 2$$

and

$$d = \frac{16}{19}$$

Therefore, the first 4 terms are

$$a$$

$$a + d$$

$$a + 2d$$

$$a + 3d$$

which give

$$2$$

$$2 + \frac{16}{19}$$

$$2 + \frac{32}{19}$$

$$2 + \frac{48}{19}$$

and the first 4 terms are

$$2,\ 2\frac{16}{19},\ 3\frac{13}{19},\ 4\frac{10}{19}$$

PROBLEMS: Find the sum of the sequence having

1. $a = 3$, $\ell = 7$, $n = 15$
2. $a = -6$, $\ell = 18$, $n = 6$
3. $a = 7$, $n = 5$, $d = 2$
4. $d = 6$, $\ell = 32$, $n = 5$
5. $a = -19$, $d = 3$, $\ell = -7$

In problems 6 and 7, find the first 4 terms if

6. $a = 6$, $\ell = 14$, $S_n = 50$
7. $n = 7$, $\ell = 20$, $S_n = 70$

ANSWERS:

1. 75
2. 36
3. 55
4. 100
5. -65
6. 6, 8, 10, 12
7. $0, 3\frac{1}{3}, 6\frac{2}{3}, 10$

GEOMETRIC SEQUENCES

A geometric sequence (or progression) is a sequence (or progression) in which each term is a multiple of any other, with a constant ratio between adjacent terms. This constant, called the common ratio, is designated by the letter r and will maintain the same value throughout the sequence.

A geometric sequence, then, may be indicated by $a, ar, ar^2, \ldots ar^{(n-1)}$, where there are n terms in the sequence. The common ratio (r) in a geometric sequence may be determined by dividing any term by its preceding term. The quotient is the common ratio.

In the sequence

$$2, 6, 18, \ldots$$

the common ratio r is 3. The first term is a and is equal to 2. This may be shown by

$$a = 2$$

$$ar = 2 \cdot 3$$

$$ar^2 = 2 \cdot 3^2$$

If there are n terms in the sequence, then the last term is

$$ar^{(n-1)}$$

Notice that if we considered this sequence to have only three terms then the last term would be

$$\ell = ar^{(n-1)}$$

$$= 2 \cdot 3^2$$

$$= 18$$

EXAMPLE: Find the next three terms in the sequence

$$3, 12, 48, \ldots$$

SOLUTION: The first term a is 3 and the ratio is

$$r = \frac{12}{3}$$

$$= 4$$

Therefore, the fourth, fifth, and sixth terms are

$$ar^{(4-1)},\ ar^{(5-1)},\ \text{and } ar^{(6-1)}$$

Then,

$$\begin{aligned} a(r)^{(4-1)} &= 3(4)^3 \\ &= 3 \cdot 64 \\ &= 192 \end{aligned}$$

and

$$\begin{aligned} a(r)^{5-1} &= 3(4)^4 \\ &= 3 \cdot 256 \\ &= 768 \end{aligned}$$

and

$$\begin{aligned} a(r)^{6-1} &= 3(4)^5 \\ &= 3 \cdot 1024 \\ &= 3072 \end{aligned}$$

EXAMPLE: Find the last term of the sequence where

$$a = 3$$

$$n = 5$$

and

$$r = 2$$

SOLUTION: Write

$$\begin{aligned} \ell &= a(r)^{(n-1)} \\ &= 3(2)^{5-1} \\ &= 3(2)^4 \\ &= 48 \end{aligned}$$

EXAMPLE: Find the first term of a sequence if the second term is 6, the third term is 24, and the fourth term is 96.

SOLUTION: We consider the last term as 96 and write

$$\ell = a(r)^{(n-1)}$$

We desire a, but we do not know r. To find r we divide any term by the preceding term; that is,

$$\frac{96}{24} = 4$$

or

$$\frac{24}{6} = 4$$

and find r to be 4.

Substitution yields

$$96 = a(4)^{(4-1)}$$

where n is 4 because 96 is the fourth term. Then,

$$\begin{aligned} 96 &= a(4)^3 \\ &= a(64) \end{aligned}$$

and

$$\begin{aligned} a &= \frac{96}{64} \\ &= \frac{3}{2} \end{aligned}$$

The sequence is

$$\frac{3}{2},\ 6,\ 24,\ 96.$$

PROBLEMS: Write the first three terms of each sequence if

1. $a = 2,\ r = 5$

2. $a = -3,\ r = \frac{1}{2}$

3. $a = 1,\ r = .01$

Find the last term of each sequence

4. $a = 7,\ n = 5,\ r = 2$

5. $a = \frac{1}{2},\ n = 4,\ r = \frac{1}{3}$

6. $30,\ 10,\ 3\frac{1}{3},\ \ldots$ and $n = 6$

ANSWERS:

1. 2, 10, 50
2. $-3, -\frac{3}{2}, -\frac{3}{4}$
3. 1, .01, .0001
4. 112
5. $\frac{1}{54}$
6. $\frac{10}{81}$

GEOMETRIC MEANS

In the sequence 5, 15, 45, 135, the terms 15 and 45 occur between the first term 5 and the last term 135 and are designated the means. Generally, the terms which occur between two given terms are called the means.

If we are given the first term (a) and the last term (ℓ) in a sequence of n terms, then there are (n - 2) means between a and ℓ. There can be any number of means between two given terms of a sequence, depending on the common ratio between adjacent terms.

In order that the means between terms in a sequence may be inserted, the common ratio must be known.

To find the means between two terms of a sequence we use the formula

$$\ell = a(r)^{n-1}$$

EXAMPLE: Insert two means between 3 and 24.

SOLUTION: We consider 3 the first term and 24 the last term and write

$$\underline{3}\ \_\ \_\ \underline{24}$$

There are four terms in the sequence and we write

$$\ell = a(r)^{n-1}$$

By substitution

$$24 = 3(r)^3$$

$$\frac{24}{3} = r^3$$

$$r^3 = 8$$

$$r = 2$$

Now find the means underlined

$$a,\ \underline{ar},\ \underline{ar^2},\ ar^3$$

to be

$$ar = 3 \cdot 2$$

$$= 6$$

and

$$ar^2 = 3(2)^2$$

$$= 3 \cdot 4$$

$$= 12$$

EXAMPLE: Insert a mean between $\frac{1}{2}$ and $\frac{9}{32}$.

SOLUTION: We consider there are three terms and write

$$a = \frac{1}{2}$$

$$\ell = \frac{9}{32}$$

$$n = 3$$

therefore,

$$\ell = a(r)^{n-1}$$

$$\frac{9}{32} = \frac{1}{2}(r)^2$$

$$r^2 = \frac{9}{32} \div \frac{1}{2}$$

$$r^2 = \frac{9}{16}$$

$$r = \pm\sqrt{\frac{9}{16}}$$

$$r = \frac{3}{4} \text{ or } -\frac{3}{4}$$

We find the mean we desire is the second term and write

$$ar = \frac{1}{2} \cdot \frac{3}{4}$$

$$= \frac{3}{8}$$

and the sequence is

$$\frac{1}{2}, \frac{3}{8}, \frac{9}{32}$$

or

$$ar = \frac{1}{2}\left(-\frac{3}{4}\right)$$

$$= -\frac{3}{8}$$

and the sequence is

$$\frac{1}{2}, -\frac{3}{8}, \frac{9}{32}$$

In cases where only one mean is required, the following may be used. We know the common ratio may be found by dividing any term by the preceding term; that is, if the sequence is a, m, ℓ, then the common ratio is either

$$\frac{m}{a} = r$$

or

$$\frac{\ell}{m} = r$$

therefore,

$$\frac{m}{a} = \frac{\ell}{m}$$

and

$$m^2 = a\ell$$

$$m = \sqrt{a\ell} \text{ or } -\sqrt{a\ell}$$

In the previous example where we wanted to find the one mean between $\frac{1}{2}$ and $\frac{9}{32}$ we could have written

$$m = \pm\sqrt{a\ell}$$

$$= \pm\sqrt{\frac{1}{2} \cdot \frac{9}{32}}$$

$$= \pm\sqrt{\frac{9}{64}}$$

$$= \frac{3}{8} \text{ or } -\frac{3}{8}$$

PROBLEMS: Insert the indicated number of means in the sequences.

1. Two, between 3 and 24
2. Two, between $\frac{1}{2}$ and $\frac{1}{54}$
3. One, between 4 and 9
4. Three, between x^2 and x^{10}
5. Four, between -5 and $-\frac{5}{243}$

ANSWERS:

1. 6, 12
2. $\frac{1}{6}, \frac{1}{18}$
3. 6 or -6
4. x^4, x^6, x^8 or $-x^4, x^6, -x^8$
5. $-\frac{5}{3}, -\frac{5}{9}, -\frac{5}{27}, -\frac{5}{81}$

GEOMETRIC SERIES

When we add all the terms of a sequence, we call this indicated sum a series. We use the symbol S_n to designate the indicated sum of n terms of a sequence. To derive a formula for S_n we may write the terms of a series as

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

then multiply each term of the series by (-r) to obtain

$$-rS_n = -ar - ar^2 - ar^3 - \ldots - ar^{n-1} - ar^n$$

and combine the two equations as follows:

$$\begin{array}{rl} S_n = & a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \\ -rS_n = & \quad - ar - ar^2 - \cdots - ar^{n-2} - ar^{n-1} - ar^n \\ \hline S_n - rS_n = & a \qquad\qquad\qquad\qquad\qquad\qquad - ar^n \end{array}$$

$$S_n(1 - r) = a - ar^n$$

$$S_n = \frac{a - ar^n}{1 - r}$$

$$= \frac{a(1 - r^n)}{1 - r}, \; r \neq 1$$

EXAMPLE: Find the sum of six terms in the series whose first term is 3 and whose common ratio is 2.

SOLUTION: We know

$$a = 3$$

$$r = 2$$

and

$$n = 6$$

therefore,

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$= \frac{3(1 - 2^6)}{1 - 2}$$

$$= \frac{3(1 - 64)}{-1}$$

$$= \frac{3(-63)}{-1}$$

$$= \frac{-189}{-1}$$

$$= 189$$

In cases where we know the last term, the first term, and the common ratio and desire S_n we could use

$$S_n = \frac{a - ar^n}{1 - r}$$

but we would first have to determine n.

EXAMPLE: Find the sum of a series if

$$a = 3, \; r = 4, \text{ and } \ell = 192.$$

SOLUTION: To find n we write

$$\ell = ar^{n-1}$$

and by substitution

$$192 = (3)(4)^{n-1}$$

$$\frac{192}{3} = 4^{n-1}$$

$$64 = 4^{n-1}$$

then

$$64 = 4^3$$

and

$$4^3 = 4^{n-1}$$

Therefore,

$$3 = n - 1$$

$$n = 4$$

Now

$$S_n = \frac{a - ar^n}{1 - r}$$

$$= \frac{3 - 3(4)^4}{1 - 4}$$

$$= \frac{3 - 768}{-3}$$

$$= \frac{-765}{-3}$$

$$= 255$$

In order to decrease the number of operations in the previous example we may write

$$S_n = \frac{a - ar^n}{1 - r}$$

and

$$ar^n = r(ar^{n-1})$$

Therefore,

$$S_n = \frac{a - r(ar^{n-1})}{1 - r}$$

but

$$ar^{n-1} = \ell$$

then

$$S_n = \frac{a - r\ell}{1 - r}$$

The solution to the previous problem would be

$$\begin{aligned} S_n &= \frac{3 - 4(192)}{1 - 4} \\ &= \frac{3 - 768}{-3} \\ &= \frac{-765}{-3} \\ &= 255 \end{aligned}$$

PROBLEMS: Find S_n in the following series if

1. $a = 3$, $r = 5$, and $n = 4$
2. $a = \frac{1}{2}$, $r = \frac{1}{3}$, and $n = 3$
3. $a = -\frac{1}{3}$, $r = 6$, and $n = 4$
4. $a = 4$, $r = 3$, and $\ell = 324$
5. $a = \frac{2}{3}$, $r = -\frac{1}{2}$, and $\ell = -\frac{1}{12}$
6. $a = \frac{5}{3}$, $r = 3$, and $\ell = 32{,}805$

ANSWERS:

1. 468
2. $\frac{13}{18}$
3. $-86\frac{1}{3}$
4. 484
5. $\frac{5}{12}$
6. $49{,}206\frac{2}{3}$

INFINITE SERIES

As previously discussed, a series is the indicated sum of the terms of a sequence. If the number of terms of a series is unlimited, the series is said to be infinite; that is,

$$1 + 2 + 4 + \cdots$$

is an infinite series and

$$1 + 2 + 4 + \cdots + n$$

is a finite series because there is a finite number of terms.

When we desire the sum of a geometric series such as

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

and we know the number of terms, we use the formula

$$S_n = \frac{a - ar^n}{1 - r}$$

This formula may be written as

$$\begin{aligned} S_n &= \frac{a - ar^n}{1 - r} \\ &= \frac{a}{1 - r} - \frac{ar^n}{1 - r} \end{aligned}$$

If we increase the number of terms desired, notice that the second term

$$\frac{ar^n}{1 - r}$$

becomes larger if $|r| > 1$ and becomes smaller if $|r| < 1$.

When the number of terms of a series continues indefinitely, the series is an infinite series. Therefore, in

$$\frac{ar^n}{1 - r}$$

as $n \to \infty$ the term goes to ∞ if $|r| > 1$ and the term goes to zero if $|r| < 1$. When the term

$$\frac{ar^n}{1 - r}$$

goes to ∞, the sum of the series is not defined. However, if this term goes to zero, we may write the sum of the series as

$$\lim_{n \to \infty} S_n = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

$$= \frac{a}{1 - r} - 0$$

$$= \frac{a}{1 - r}$$

which is the sum of an infinite series when $|r| < 1$. We designate the limit of the sum of an infinite series as

$$S = \frac{a}{1 - r}$$

EXAMPLE: Find the sum of the infinite series

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \cdots$$

SOLUTION: Determine that

$$r = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$$

Then

$$S = \frac{a}{1 - r}$$

$$= \frac{\frac{1}{3}}{1 - \frac{1}{2}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{2}}$$

$$= \frac{2}{3}$$

which is the limiting value of the infinite series.

EXAMPLE: Find the sum of the infinite series

$$7 + \frac{7}{5} + \frac{7}{25} + \cdots$$

SOLUTION: Determine that

$$r = \frac{\frac{7}{5}}{7}$$

$$= \frac{1}{5}$$

then

$$S = \frac{a}{1 - r}$$

$$= \frac{7}{1 - \frac{1}{5}}$$

$$= \frac{7}{\frac{4}{5}}$$

$$= \frac{35}{4}$$

$$= 8\frac{3}{4}$$

PROBLEMS: Find S in the following:

1. $1 + \frac{2}{3} + \frac{4}{9} + \cdots$
2. $6 + 2 + \frac{2}{3} + \cdots$
3. $.1 + .01 + .001 + \cdots$
4. $\sqrt{2} + 1 + \frac{\sqrt{2}}{2} + \cdots$

ANSWERS:

1. 3
2. 9

3. $\frac{1}{9}$

4. $2\sqrt{2} + 2$

THE n^{th} TERM

In cases where we are given the n^{th} term of a series, it is relatively easy to find other terms from the symbolic definition of the n^{th} term.

EXAMPLE: If the n^{th} term of a series is given by

$$\frac{n}{2n + 1}$$

find the first three terms.

SOLUTION: To determine the first term replace n by 1 and for the second term replace n by 2, etc.; that is, the first term is

$$\frac{n}{2n + 1} = \frac{1}{2(1) + 1}$$

$$= \frac{1}{3}$$

and the second term is

$$\frac{2}{2(2) + 1}$$

$$= \frac{2}{5}$$

and the third term is

$$\frac{3}{2(3) + 1}$$

$$= \frac{3}{7}$$

To perform the converse of this type problem, that is, to find the n^{th} term of a given series, is quite different because there are no set rules which may be applied. Also, there may be many formulas which express the n^{th} term of a series.

EXAMPLE: Find the n^{th} term of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$$

SOLUTION: The numerator of the terms remains the same and is 1. The denominator follows a regular pattern of increasing by 2 for each term.

If we write

term	1	2	3	4
numerator	1	1	1	1
denominator	2	4	6	8

we see that each term's denominator may be written as

$$2 \cdot n$$

Therefore, the n^{th} term for the series is

$$\frac{1}{2n}$$

and the series may be designated as

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n} + \cdots$$

PROBLEMS: Find the first 3 terms of the series whose n^{th} term is given by

1. n^2

2. $\frac{1}{n^2 + 3}$

3. $\frac{n^2}{n + 1}$

ANSWERS:

1. $1 + 4 + 9 + \cdots$

2. $\frac{1}{4} + \frac{1}{7} + \frac{1}{12} + \cdots$

3. $\frac{1}{2} + \frac{4}{3} + \frac{9}{4} + \cdots$

PROBLEMS: Find the n^{th} term of the series

1. $\frac{1}{6} + \frac{2}{7} + \frac{3}{8} + \cdots$

2. $1 + \frac{1}{8} + \frac{1}{27} + \cdots$

3. $1 + \frac{3}{4} + \frac{5}{9} + \cdots$

ANSWERS:

1. $\frac{n}{n+5}$

2. $\frac{1}{n^3}$

3. $\frac{2n-1}{n^2}$

CONVERGENCE

If we find the sum of the first n terms of an infinite series approaches a finite value as $n \to \infty$, then we say the series is convergent. If a series is not convergent, then we say it is divergent.

EXAMPLE: Is the infinite series

$$1 + \frac{1}{3} + \frac{1}{9} + \cdots \text{ convergent?}$$

SOLUTION: Write

$$\lim_{n \to \infty} S_n = S = \frac{a}{1-r}$$

and

$$r = \frac{1}{3}$$

$$a = 1$$

then

$$S = \frac{1}{1 - \frac{1}{3}}$$

$$= \frac{1}{\frac{2}{3}}$$

$$= \frac{3}{2}$$

The limit of the sum of n terms as $n \to \infty$ approaches $\frac{3}{2}$, therefore the series is convergent.

EXAMPLE: Is the infinite series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \text{ convergent?}$$

SOLUTION: Write

$$S = \frac{a}{1-r}$$

because this is an infinite geometric series where

$$a = \frac{1}{2}$$

$$r = \frac{1}{2}$$

then

$$S = \frac{\frac{1}{2}}{1 - \frac{1}{2}}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}}$$

$$= 1$$

The sum has a limit, therefore the series is convergent on 1.

EXAMPLE: Is the infinite series

$$3 + 6 + 12 + \cdots \text{ convergent?}$$

SOLUTION: Find that

$$a = 3$$

and

$$r = \frac{6}{3} = 2$$

Now,

$$|r| > 1$$

therefore

$$\lim_{n \to \infty} S_n = \frac{a - ar^n}{1-r}$$

and the sum of the series goes to ∞ and is not defined.

EXAMPLE: Is the series

$$1 + 3 + 5 + \cdots \text{ convergent?}$$

SOLUTION: Find that the series is arithmetic and

$$a = 1$$

and the n^{th} term is

$$(2n - 1)$$

Consider the n^{th} term as the last term and write

$$\begin{aligned} S_n &= \frac{n}{2}(a + \ell) \\ &= \frac{n}{2}[1 + (2n - 1)] \\ &= n^2 \end{aligned}$$

Then

$$\lim_{n \to \infty} S_n = \infty$$

PROBLEMS: Determine whether the following series are convergent or divergent.

1. $3 + 6 + 9 + \cdots + 3^n + \cdots$
2. $1 + 3 + 9 + \cdots + 3^{n-1} + \cdots$
3. $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^{n-1}} + \cdots$

ANSWERS:

1. Divergent
2. Divergent
3. Convergent

If a series is convergent, its n^{th} term must have zero as its limit. But, if the n^{th} term of a series has a limit of zero as $n \to \infty$, this does not mean the series is convergent. If the n^{th} term of a series does not have zero as a limit, then the series is divergent. That is, if the limit of the n^{th} term is zero, as $n \to \infty$, the series may or may not be convergent.

EXAMPLE: Determine if the series

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \cdots + \cdots + \frac{1}{3^n} + \cdots \text{ is convergent}$$

SOLUTION: Examine the last term and find that

$$\lim_{n \to \infty} \frac{1}{3^n} = 0$$

This is a necessary condition, but we must investigate the series further. We find that the series is geometric with

$$|r| < 1$$

Therefore, we conclude the series converges.

EXAMPLE: Determine if the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots \text{ is convergent.}$$

NOTE: This is a harmonic series. A harmonic series is a series whose reciprocals form an arithmetic series.

SOLUTION: Investigation of the n^{th} term indicates that

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

We now expand the series by writing

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{n} + \cdots$$

If we group terms as follows:

$$\begin{aligned} &1 \\ &+ \frac{1}{2} \\ &+ \frac{1}{3} + \frac{1}{4} \\ &+ \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\ &+ \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \end{aligned}$$

we find

$$1 > \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{1}{2}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{533}{840} > \frac{1}{2}$$

$$\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} > \frac{1}{2}$$

and if we continue to group terms after the second term in groups of 2, 4, 8, 16, 32, . . . we find the sum of each group is greater than $\frac{1}{2}$. There is an unlimited number of groups, therefore, the limit of the sum

$$\lim_{n \to \infty} S_n = \infty$$

and the series is divergent.

The two previous examples indicate how convergence or divergence of a series is determined. We will use four types of series as reference; that is,

$$a + ar + ar^2 + \cdots + ar^{(n-1)} + \cdots \qquad (1)$$

is convergent, $|r| < 1$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

or

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots \qquad (2)$$

is convergent

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \qquad (3)$$

is divergent, and

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots \qquad (4)$$

is convergent, $p > 1$; and is divergent, $p \leq 1$

TEST FOR CONVERGENCE BY COMPARISON

If we know that the series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is convergent and we wish to know if the series

$$B_1 + B_2 + B_3 + \cdots + B_n + \cdots$$

is convergent, we compare the two series term by term. If we find that every term a_i is greater than or equal to every term B_i, that is,

$$a_i \geq B_i$$

then the series under investigation is convergent.

This is because the limit of the sum of terms of the reference series is greater than the limit of the sum of the series under investigation.

EXAMPLE: Test for convergence the series

$$\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots + \frac{1}{5^n} + \cdots$$

SOLUTION: We use the reference series (2) and write

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots \qquad (2)$$

and

$$\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots + \frac{1}{5^n} + \cdots$$

In term by term comparison we find

$$\frac{1}{2} > \frac{1}{5}$$

$$\frac{1}{2^2} > \frac{1}{25}$$

$$\frac{1}{2^3} > \frac{1}{125}$$

$$\frac{1}{2^n} > \frac{1}{5^n}$$

Since the reference series is convergent, the series under investigation is convergent.

If we desire to test a series for divergence by comparison, we use a reference series which is divergent. Then, if each term of the series under investigation is greater than or equal to the reference series, it too is divergent.

EXAMPLE: Test for divergence the series

$$1 + 3 + 5 + \cdots + (2n - 1) + \cdots$$

SOLUTION: Use the reference series (3); that is,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

and compare, term by term, which yields

$$1 \geq 1$$

$$3 \geq \frac{1}{2}$$

$$5 \geq \frac{1}{3}$$

and

$$2n - 1 \geq \frac{1}{n}$$

then the reference series is divergent, and the series under investigation, term by term, is equal to or greater than the reference and is therefore divergent.

RATIO TEST FOR CONVERGENCE

The ratio test is limited to series where all terms are positive. In this test we must write the nth term and the (n + 1)th term and find the limit of the ratio of these two terms. That is,

$$\lim_{n \to \infty} \frac{t_{(n+1)}}{t_n} = r$$

where $t_{(n+1)}$ is the (n + 1)th term.

If $|r| < 1$ the series is convergent and if $|r| > 1$ the series is divergent. If $|r| = 1$, the test fails because the series could be either convergent or divergent.

EXAMPLE: Test for convergence the series

$$10 + \frac{10^2}{1 \cdot 2} + \frac{10^3}{1 \cdot 2 \cdot 3} + \cdots + \frac{10^n}{1 \cdot 2 \cdot 3 \cdots n} + \cdots$$

SOLUTION: Write the term t_n as

$$\frac{10^n}{1 \cdot 2 \cdot 3 \cdots n}$$

and the term $t_{(n+1)}$ as

$$\frac{10^{n+1}}{1 \cdot 2 \cdot 3 \cdots (n + 1)}$$

The ratio is

$$\frac{\dfrac{10^{n+1}}{1 \cdot 2 \cdot 3 \cdots (n + 1)}}{\dfrac{10^n}{1 \cdot 2 \cdot 3 \cdots n}}$$

Then,

$$\frac{10^{(n+1)}}{1 \cdot 2 \cdot 3 \cdots (n + 1)} \cdot \frac{1 \cdot 2 \cdot 3 \cdots n}{10^n}$$

$$= \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots (n + 1)} \cdot \frac{10^{(n+1)}}{10^n}$$

$$= \frac{n!}{(n + 1)!} \cdot \frac{10^{(n+1)}}{10^n}$$

$$= \frac{n!}{(n + 1)!} \cdot \frac{10^{(n+1)-n}}{1}$$

$$= \frac{n!}{(n + 1)!} \cdot \frac{10}{1}$$

$$= \frac{n!}{n!(n + 1)} \cdot \frac{10}{1}$$

$$= \frac{10}{(n + 1)}$$

and

$$\lim_{n \to \infty} \frac{t_{(n+1)}}{t_n}$$

$$= \lim_{n \to \infty} \frac{10}{n + 1}$$

$$= 0$$

Therefore, the $|r| < 1$ and the series is convergent.

EXAMPLE: Test for convergence the series

$$\frac{2}{1^3} + \frac{2^2}{2^3} + \frac{2^3}{3^3} + \cdots + \frac{2^n}{n^3} + \cdots$$

SOLUTION: The t_n term is

$$\frac{2^n}{n^3}$$

and the $t_{(n+1)}$ term is

$$\frac{2^{n+1}}{(n+1)^3}$$

therefore,

$$\frac{\dfrac{2^{n+1}}{(n+1)^3}}{\dfrac{2^n}{n^3}}$$

$$= \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n}$$

$$= \frac{2^{n+1}}{2^n} \cdot \frac{n^3}{(n+1)^3}$$

$$= \frac{2}{1} \cdot \frac{n^3}{(n+1)^3}$$

$$= \frac{2n^3}{(n+1)^3}$$

$$= 2\left(\frac{n}{n+1}\right)^3$$

and

$$2\left(\frac{n}{n+1}\right)^3$$

$$= 2\left[\frac{n}{n+1} \cdot \left(\frac{\frac{1}{n}}{\frac{1}{n}}\right)\right]^3$$

$$= 2\left(\frac{1}{1+\frac{1}{n}}\right)^3$$

Then,

$$\lim_{n \to \infty} 2\left(\frac{n}{n+1}\right)^3$$

$$= 2 \lim_{n \to \infty} \left(\frac{n}{n+1}\right)^3$$

$$= 2 \lim_{n \to \infty} \left(\frac{1}{1+\frac{1}{n}}\right)^3$$

$$= 2\,(1)^3$$

$$= 2$$

$$= r$$

and

$$|r| > 1$$

therefore, the series diverges.

PROBLEMS: Test for convergence by the comparison test:

1. $\frac{1}{1^2+2} + \frac{1}{2^2+2} + \frac{1}{3^2+2} + \cdots + \frac{1}{n^2+2} + \cdots$

Test for convergence by the ratio test:

2. $\frac{1}{1(3)} + \frac{1}{2(3)^2} + \frac{1}{3(3)^3} + \cdots + \frac{1}{n(3)^n} + \cdots$

3. $\frac{4}{1^3} + \frac{4^2}{2^3} + \frac{4^3}{3^3} + \cdots + \frac{4^n}{n^3} + \cdots$

ANSWERS:

1. Convergent
2. Convergent
3. Diverges

CHAPTER 2

MATHEMATICAL INDUCTION AND THE BINOMIAL THEOREM

In this chapter we will investigate a method of proof called mathematical induction and then use mathematical induction to verify the binomial theorem for all positive integral values of n. We will also consider this theorem for fractional and negative values of n.

MATHEMATICAL INDUCTION

Mathematical induction is a proof that follows the idea that if we have a stairway of infinite steps and if we know we can take the first step and also that we can take a single step from any other step, we can, by taking steps one at a time, climb the stairway.

This proof is separated into two parts. First, we prove we can take the first step. Second, we assume we can reach a particular step, then we prove we can take one step from that particular step; therefore, we can climb the stairway.

To illustrate proof by mathematical induction we will prove that the sum of consecutive even integers in a series is equal to n (n + 1), where n represents the number of terms. The series is

$$2 + 4 + 6 + \cdots$$

We want to prove that

$$2 + 4 + 6 + \cdots + 2n = n\,(n + 1)$$

When n is 1 the formula yields

$$1\,(1 + 1)$$

$$= 2$$

This is true since the first term is shown to be 2. We check the formula when n is 2, although this is not necessary, and find that

$$2\,(2 + 1)$$

$$= 6$$

The sum of the first two terms is

$$2 + 4 = 6$$

which verifies the formula when n equals two.

We could verify the formula for as many values of n as we desire but this would not prove the formula for every value of n. We must now show that if the formula holds for the case where n equals K, then it holds for n equals (K + 1). When n equals K in

$$2 + 4 + 6 + \cdots + 2n = n\,(n + 1)$$

we have

$$2 + 4 + 6 + \cdots + 2K = K\,(K + 1)$$

This we assume to be true.
Then, when n equals (K + 1) we write

$$\begin{aligned} 2 + 4 + 6 + \cdots + 2(K + 1) &= (K + 1)\,(K + 1 + 1) \\ &= (K + 1)\,(K + 2) \end{aligned}$$

To show that this is true we add the $(K + 1)^{th}$ term to both sides of our assumed equality

$$2 + 4 + 6 + \cdots + 2K = K\,(K + 1)$$

which gives

$$2 + 4 + 6 + \cdots + 2K + 2(K+1) = K(K+1) + 2(K+1)$$

In order to show that this is equal to

$$(K + 1)(K + 2)$$

we write

$$\begin{aligned} K(K + 1) + 2(K + 1) &= (K + 1)\,(K + 2) \\ K^2 + K + 2K + 2 &= (K + 1)\,(K + 2) \\ K^2 + 3K + 2 &= (K + 1)\,(K + 2) \\ (K + 1)(K + 2) &= (K + 1)\,(K + 2) \end{aligned}$$

EXAMPLE: Use mathematical induction to prove that for all positive integral values,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

SOLUTION: First step—Verify that this is true for $n = 1$. Substitute 1 for n and find that

$$\begin{aligned} n^2 &= 1^2 \\ &= 1 \end{aligned}$$

which is the "sum" of the first term.

Second step—Assume the statement is true for $n = K$, then show it is true for $n = K + 1$, the next greater term than $n = K$.

When we assume the statement true for $n = K$, we may write

$$1 + 3 + 5 + \cdots + (2K - 1) = K^2$$

We now say that if $n = K + 1$ we may substitute $K + 1$ for n in the original statement and write

$$1 + 3 + 5 + \cdots + [2(K + 1) - 1] = (K + 1)^2$$

We must now verify that this is identical to adding the next term to both members of

$$1 + 3 + 5 + \cdots + (2K - 1) = K^2$$

We do this and find

$$1 + 3 + 5 + \cdots + (2K - 1) + (2K + 1) = K^2 + 2K + 1$$

We now verify that

$$K^2 + 2K + 1 = (K + 1)^2$$

Factoring the left member we find

$$\begin{aligned} K^2 + 2K + 1 &= (K + 1)(K + 1) \\ &= (K + 1)^2 \end{aligned}$$

which completes the verification.

We know that the original statement is true for $n = 1$. We proceed by letting $n = K = 1$. We find this is true. If we let $K = 2$, we find that

$$\begin{aligned} 1 + 3 &= 2^2 \\ &= 4 \end{aligned}$$

and when $K = 2$, we find that $K + 1 = 3$ then

$$\begin{aligned} 1 + 3 + 5 &= (K + 1)^2 \\ &= 3^2 \\ &= 9 \end{aligned}$$

Therefore, we may reason it is true for any positive integral value of n.

In general, to prove the validity of a given formula by the use of mathematical induction, we use two steps:

(1) Verify the given formula for $n = 1$.

(2) Assume the formula holds for $n = K$, then prove it is valid for $n = K + 1$ or the next larger value of n.

PROBLEMS: Prove by mathematical induction that the following series are valid for any positive integral value of n. Show steps (1) and (2).

1. $3 + 6 + 9 + \cdots + 3n = \dfrac{3n(n + 1)}{2}$
2. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$
3. $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n - 1)}{2}$

ANSWERS:

1. (1) $n = 1$ then $\dfrac{3(1)(1 + 1)}{2} = 3$ (first term)

(2) $n = K$ then $\dfrac{3K(K + 1)}{2}$

$n = K + 1$ then $\dfrac{3(K + 1)(K + 1 + 1)}{2}$

$$= \frac{3(K + 1)(K + 2)}{2}$$

and

$$\begin{aligned} \frac{3K(K+1)}{2} + 3(K+1) &= \frac{3K(K+1) + 6(K+1)}{2} \\ &= \frac{3(K + 1)(K + 2)}{2} \end{aligned}$$

2. (1) $n = 1$ then $\dfrac{1(1 + 1)(2 + 1)}{6} = 1$

(2) $n = K$ then $\dfrac{K(K + 1)(2K + 1)}{6}$

$n = K + 1$ then

$$\frac{(K + 1)(K + 1 + 1)(2K + 2 + 1)}{6}$$

$$= \frac{(K + 1)(K + 2)(2K + 3)}{6}$$

and

$$\frac{K(K + 1)(2K + 1)}{6} + (K + 1)^2$$

$$= \frac{K(K + 1)(2K + 1) + 6(K + 1)^2}{6}$$

$$= (K + 1)\left[\frac{2K^2 + 7K + 6}{6}\right]$$

$$= \frac{(K + 1)(K + 2)(2K + 3)}{6}$$

3. (1) $n = 1$ then $\dfrac{1(3 - 1)}{2} = 1$

(2) $n = K$ then $\dfrac{K(3K - 1)}{2}$

$n = K + 1$ then $\dfrac{(K + 1)(3K + 3 - 1)}{2}$

$$= \frac{(K + 1)(3K + 2)}{2}$$

and

$$\frac{K(3K - 1)}{2} + (3K + 1) = \frac{K(3K - 1) + 2(3K + 1)}{2}$$

$$= \frac{3K^2 - K + 6K + 2}{2}$$

$$= \frac{3K^2 + 5K + 2}{2}$$

$$= \frac{(K + 1)(3K + 2)}{2}$$

BINOMIAL THEOREM

The binomial theorem enables us to write any power of a binomial in the form of a sequence. This theorem is very useful in the study of probability and statistics. It is also useful in many other fields of mathematics.

EXPANSION

We use the binomial $(x + y)$ to indicate a general binomial, and to expand this binomial we raise it to increasing powers. That is, $(x + y)^n$ where n takes on the values 1, 2, 3, ... It is rather simple to raise the binomial to

$(x + y)^0 = 1, \qquad n = 0$

$(x + y)^1 = x + y, \qquad n = 1$

$(x + y)^2 = x^2 + 2xy + y^2, \qquad n = 2$

When we increase the value of n to 3, 4, 5, . . . we find it easier to use repeated multiplication. This results in

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

and

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

If we consider the expansion $(x + y)^5$, we may determine the following:

(1) There are $n + 1$ terms in the expansion.

(2) x is the only variable in the first term and y is the only variable in the last term.

(3) In both the first and last term the exponent is n.

(4) As we move from left to right the exponent of x decreases by one and the sum of the exponents of each term is equal to n.

(5) The numerical coefficient of each term is determined from the term which precedes it by using the rule: The product of the exponent of x and the numerical coefficient, divided by the number which designates the position of the term, gives the value of the coefficient.

(6) There is symmetry about the middle term or terms of the numerical coefficients.

EXAMPLE: Write the expansion of $(x + y)^6$.

SOLUTION: (1) We know there are $n + 1$ or 7 terms.

(2) and (3) The first term is x^6 and the last term is y^6.

(4) The terms with their exponents but without their coefficients are

$$x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6$$

The numerical coefficient of the second term (determined by the first term) is

$$\frac{6 \cdot 1}{1} = 6$$

Then, we have $x^6 + \underline{6x^5y}$

The numerical coefficient of the third term (determined by the second term, $6x^5y$) is

$$\frac{5 \cdot 6}{2} = 15$$

Then, we have

$$x^6 + 6x^5y + \underline{15x^4y^2}$$

The numerical coefficient of the fourth term is

$$\frac{15 \cdot 4}{3} = 20$$

Then, we have $x^6 + 6x^5y + 15x^4y^2 + \underline{20x^3y^3}$

This process is continued to find

$$x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

EXAMPLE: Write the expansion of $(2x - 3y)^4$

SOLUTION: In this case we consider $(2x - 3y)^4$ to be

$$[(2x) + (-3y)]^4$$

We write the terms without coefficients as

$$(2x)^4 + (2x)^3(-3y) + (2x)^2(-3y)^2$$
$$+ (2x)(-3y)^3 + (-3y)^4$$

then determine the numerical coefficients as

$$(2x)^4 + 4(2x)^3(-3y) + 6(2x)^2(-3y)^2$$
$$+ 4(2x)(-3y)^3 + (-3y)^4$$

and carry out the multiplication indicated to find

$$16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 108y^4$$

EXAMPLE: Evaluate $(1 + 0.05)^5$ to the nearest hundredth.

SOLUTION: Write

$$(1 + 0.05)^5 = 1^5 + 5(1)^4(0.05)$$
$$+ 10(1)^3(0.05)^2 + 10(1)^2(0.05)^3$$
$$+ 5(1)^1(0.05)^4 + (0.05)^5$$

Term by term the values are

$$1^5 = 1$$

$$5(1)^4(0.05) = 0.25$$

$$10(1)^3(0.05)^2 = 0.025$$

$$10(1)^2(0.05)^3 = 0.00125$$

$$5(0.05)^4 = 0.00003625$$

$$(0.05)^5 = 0.0000003625$$

We are concerned only with hundredths; therefore, we add only the first four terms and find the sum to be

$$1.27625$$

which rounds to

$$1.28$$

For all positive values of n, the expansion of a binomial $(x + y)^n$ may be accomplished by following the previous rules indicated; that is,

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n - 1)x^{n-2}y^2}{1 \cdot 2}$$
$$+ \frac{n(n - 1)(n - 2)x^{n-3}y^3}{1 \cdot 2 \cdot 3} + \cdots + y^n$$

PROBLEMS: Write the expansion of the following:

1. $(x + 3y)^3$
2. $(2x - y)^4$
3. $(n + 3)^5$

ANSWERS:

1. $x^3 + 9x^2y + 27xy^2 + 27y^3$

2. $16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$

3. $n^5 + 15n^4 + 90n^3 + 270n^2 + 405n + 243$

PROBLEMS: Evaluate the following to the nearest hundredth.

1. $(1 + 0.01)^6$

2. $(1.03)^5$ or $(1 + 0.03)^5$

ANSWERS:

1. 1.06

2. 1.16

GENERAL TERM OF $(x + y)^n$

We consider the general term of $(x + y)^n$ as the r^{th} term. When we expand $(x + y)^n$ we have

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)x^{n-2}y^2}{1 \cdot 2}$$

$$+ \frac{n(n-1)(n-2)x^{n-3}y^3}{1 \cdot 2 \cdot 3} + \cdots + y^n$$

Notice that if we consider the term

$$\frac{n(n-1)(n-2)(n-3)x^{n-4}y^4}{1 \cdot 2 \cdot 3 \cdot 4}$$

as the r^{th} term, it is really the fifth term. The coefficient is

$$\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

which is really

$$\frac{n(n-1) \cdots (n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

If this is the r^{th} term, then, in the numerator

$$n - 3 = n - (r - 2)$$

$$= n - r + 2$$

and in the denominator

$$4 = r - 1$$

therefore,

$$\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

is equal to

$$\frac{n(n-1)(n-2) \cdots (n-r+2)}{1 \cdot 2 \cdot 3 \cdots (r-1)}$$

and the exponents are

$$x^{n-4}y^4$$

or

$$x^{n-(r-1)}y^{(r-1)} = x^{n-r+1}y^{r-1}$$

Therefore, the r^{th} term where $r = 1, 2, 3, \ldots$ is

$$\frac{n(n-1)(n-2)(n-3) \cdots (n-r+2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (r-1)} x^{n-r+1}y^{r-1}$$

At this point, the binomial formula holds for all positive integral values of n. Later, we will prove this to be true.

EXAMPLE: Find the 4^{th} term in the expansion of $(x + y)^8$.

SOLUTION: Write

$$n = 8$$

$$r = 4$$

then,

$$\frac{n(n-1)(n-2)(n-3) \cdots (n-r+2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (r-1)} x^{n-r+1}y^{r-1}$$

$$= \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} x^{8-4+1}y^{4-1}$$

$$= \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} x^5y^3$$

$$= 56x^5y^3$$

EXAMPLE: Find the 3^{rd} term in the expansion of $(a - 3b)^5$.

SOLUTION: Write

$$n = 5$$

$$r = 3$$

and let

$$x = a$$

and

$$y = -3b$$

Then, using the binomial formula, find that

$$\frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}\, x^{n-r+1}y^{r-1}$$

$$= \frac{5\cdot 4}{1\cdot 2}(a)^3(-3b)^2$$

$$= 10a^3(-3b)^2$$

$$= 90a^3b^2$$

PROBLEMS: Find the indicated term of the following by using the binomial formula:

1. 4^{th} term of $(x + y)^9$
2. 9^{th} term of $(x + y)^{12}$
3. 3^{rd} term of $(a^2 + B^2)^6$
4. 5^{th} term of $(2x - 3y)^7$

ANSWERS:

1. $84x^6y^3$
2. $495x^4y^8$
3. $15(a^2)^4(B^2)^2$ or $15a^8B^4$
4. $35(2x)^3(-3y)^4$ or $22680x^3y^4$

EXPANSION OF $(x + y)^n$ WHEN n IS NEGATIVE OR FRACTIONAL

The expansion of $(x + y)^n$ when n is negative or fractional does not terminate and holds only if y is numerically less than x. This is known as the binomial series.

EXAMPLE: Expand $(x + y)^{-2}$ to four terms and simplify.

SOLUTION: Write

$$(x + y)^{-2} = x^{-2} - 2x^{-3}y + 3x^{-4}y^2 - 4x^{-5}y^3 + \cdots$$

$$= \frac{1}{x^2} - \frac{2y}{x^3} + \frac{3y^2}{x^4} - \frac{4y^3}{x^5} + \cdots$$

EXAMPLE: Expand $(x + y)^{\frac{1}{2}}$ to four terms and simplify.

SOLUTION: Write

$$(x + y)^{\frac{1}{2}} = x^{\frac{1}{2}} + \frac{1}{2}\cdot x^{-\frac{1}{2}}y$$

$$+ \left(-\frac{1}{8}\right)x^{-\frac{3}{2}}y^2 + \frac{1}{16}x^{-\frac{5}{2}}y^3 + \cdots$$

$$= x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}y - \frac{1}{8}x^{-\frac{3}{2}}y^2$$

$$+ \frac{1}{16}x^{-\frac{5}{2}}y^3 + \cdots$$

$$= x^{\frac{1}{2}} + \frac{y}{2x^{\frac{1}{2}}} - \frac{y^2}{8x^{\frac{3}{2}}} + \frac{y^3}{16x^{\frac{5}{2}}} + \cdots$$

EXAMPLE: Expand $(1 + y)^{-\frac{1}{2}}$ to four terms and simplify.

SOLUTION: Write

$$(1 + y)^{-\frac{1}{2}} = 1^{\frac{-1}{2}} + \left(-\frac{1}{2}\right)(1)^{\frac{-3}{2}}y + \frac{3}{8}(1)^{\frac{-5}{2}}y^2$$

$$+ \left(-\frac{5}{16}\right)(1)^{\frac{-7}{2}}y^3 + \cdots$$

$$= 1 - \frac{1}{2}y + \frac{3}{8}y^2 - \frac{5}{16}y^3 \cdots$$

The binomial expansion is a useful tool in determining a root of a number to a particular degree of accuracy.

EXAMPLE: Evaluate $\sqrt{23}$ to the nearest tenth.

SOLUTION: Write

$$\sqrt{23} = \sqrt{25 - 2}$$

$$= (25 - 2)^{\frac{1}{2}}$$

The choice of 25 and 2 is made because 25 is the nearest square to 23. Then,

$$(25 - 2)^{\frac{1}{2}} = (25)^{\frac{1}{2}} + \frac{1}{2}\,(25)^{-\frac{1}{2}}(-2)$$

$$+ \left(-\frac{1}{8}\right)(25)^{-\frac{3}{2}}(-2)^2 + \cdots$$

$$= 5 - \frac{1}{5} - \frac{1}{250} + \cdots$$

$$= 5 - 0.20 - 0.004$$

$$= 4.796$$

$$= 4.8$$

EXAMPLE: Evaluate $\sqrt[5]{35}$ to the nearest tenth.

SOLUTION:

$$\sqrt[5]{35} = \sqrt[5]{32 + 3}$$

$$= (32 + 3)^{\frac{1}{5}}$$

The choice of 32 and 3 is made because 32 is $(2)^5$ which is the nearest 5th power to 35. Then,

$$(32 + 3)^{\frac{1}{5}} = (32)^{\frac{1}{5}} + \frac{1}{5}\,(32)^{-\frac{4}{5}}(3)$$

$$+ \left(-\frac{2}{25}\right)(32)^{-\frac{9}{5}}(3)^2 + \cdots$$

$$= 2 + \frac{3}{80} - \frac{9}{6400} + \cdots$$

$$= 2 + 0.037 - 0.001$$

$$= 2.036$$

$$= 2.04$$

$$= 2.0$$

This answer may be verified by raising 2.04 to the fifth power; that is,

$$(2.04)^5 = (2 + 0.04)^5$$

$$= 2^5 + 5(2)^4(0.04) + 10(2)^3(0.04)^2$$

$$+ 10(2)^2(0.04)^3 + \cdots$$

$$= 32 + 3.20 + 0.1280 + 0.00256 + \cdots$$

$$= 35.33$$

$$\approx 35$$

PROBLEMS: Evaluate the following to the nearest tenth.

1. $\sqrt{30}$
2. $\sqrt[4]{22}$

ANSWERS:

1. 5.5
2. 2.2

PROOF BY MATHEMATICAL INDUCTION

We have shown that the binomial theorem is indicated (for all positive integral values of n) by

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n - 1)x^{n-2}y^2}{1 \cdot 2}$$

$$+ \frac{n(n - 1)(n - 2)x^{n-3}y^3}{1 \cdot 2 \cdot 3} + \cdots$$

$$+ nxy^{n-1} + y^n$$

To prove this by mathematical induction we show the two steps of the previous proof. That is, when n equals 1 the formula yields (x + y). This is obvious by inspection.

In step (2) we assume the formula is true for n equals K by writing

$$(x+y)^K = x^K + Kx^{K-1}y + \frac{K(K-1)}{1\cdot 2}x^{K-2}y^2$$

$$+ \frac{K(K-1)(K-2)x^{K-3}y^3}{1\cdot 2\cdot 3} + \cdots + Kxy^{K-1} + y^K \quad (1)$$

Then, when n = K + 1, we have

$$(x+y)^{K+1} = x^{K+1} + (K+1)x^Ky$$

$$+ \frac{K(K+1)x^{K-1}y^2}{1\cdot 2} + \frac{K(K+1)(K-1)x^{K-2}y^3}{1\cdot 2\cdot 3} + \cdots$$

$$+ (K+1)xy^K + y^{K+1} \quad (2)$$

This is what we wish to verify. In equation (1), to obtain the (K + 1) term of $(x+y)^K$, we must multiply $(x+y)^K$ by $(x+y)$ which gives $(x+y)^{K+1}$. We must also multiply the right side of this equation by (x + y) in order to maintain our equality. When we multiply the right side of equation (1) by (x + y), we have

$$(x+y)\left[x^K + Kx^{K-1}y + \frac{K(K-1)}{1\cdot 2}x^{K-2}y^2\right.$$

$$\left.+ \frac{K(K-1)(K-2)}{1\cdot 2\cdot 3}x^{K-3}y^3 + \cdots + Kxy^{K-1} + y^K\right]$$

which gives

$$x\left[x^K + Kx^{K-1}y + \frac{K(K-1)}{1\cdot 2}x^{K-2}y^2\right.$$

$$\left.+ \frac{K(K-1)(K-2)}{1\cdot 2\cdot 3}x^{K-3}y^3 + \cdots + Kxy^{K-1} + y^K\right]$$

$$+ y\left[x^K + Kx^{K-1}y + \frac{K(K-1)}{1\cdot 2}x^{K-2}y^2\right.$$

$$\left.+ \frac{K(K-1)(K-2)}{1\cdot 2\cdot 3}x^{K-3}y^3 + \cdots + Kxy^{K-1} + y^K\right]$$

By carrying out the indicated multiplication and then combining terms we have

$$x^{K+1} + (K+1)x^Ky + \frac{K(K+1)}{1\cdot 2}x^{K-1}y^2$$

$$+ \frac{K(K+1)(K-1)}{1\cdot 2\cdot 3}x^{K-2}y^3 + \cdots + (K+1)xy^K + y^{K+1}$$

which is identical to equation (2) and the validity of the theorem is proved.

PASCAL'S TRIANGLE

When we expand $(x+y)^n$ for n = 0, 1, 2, ... we find

$$(x+y)^0 = 1$$

$$(x+y)^1 = 1x + 1y$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

We could continue this indefinitely but for explanation purposes we will stop at the point where n = 4.

If we remove everything except the numerical coefficients of each term we have

n = 0,   1
n = 1,   1 1
n = 2,   1 2 1
n = 3,   1 3 3 1
n = 4,   1 4 6 4 1

whose border forms an isosceles triangle bounded by 1's on two sides. This triangle is named for Blaise Pascal who discovered it. Pascal's triangle gives the numerical coefficients of the expansion of a binomial.

Each row, after the first, is formed from the row above it and there are n + 1 terms in each row. A row is formed by writing 1 as the first term and then adding the two numbers above and nearest to the number desired; that is,

$$\begin{array}{ll} n = 0, & 1 \\ n = 1, & \underbrace{1\quad 1} \\ n = 2, & \underbrace{1\quad 2}\underbrace{\quad 1} \\ n = 3, & \underbrace{1\quad 3}\underbrace{\quad 3}\underbrace{\quad 1} \\ n = 4, & \underbrace{1\quad 4}\underbrace{\quad 6}\underbrace{\quad 4}\underbrace{\quad 1} \\ n = 5, & 1\quad 5\quad 10\quad 10\quad 5\quad 1 \end{array}$$

EXAMPLE: Find the coefficients of $(x + y)^n$ when n equals 7.

SOLUTION: Write

$$\begin{array}{lccccccccccccccc}
n = 0, & & & & & & & & 1 & & & & & & & \\
n = 1, & & & & & & & 1 & & 1 & & & & & & \\
n = 2, & & & & & & 1 & & 2 & & 1 & & & & & \\
n = 3, & & & & & 1 & & 3 & & 3 & & 1 & & & & \\
n = 4, & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
n = 5, & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
n = 6, & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
n = 7, & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

Notice there are n + 1 terms in each row.

CHAPTER 3

DESCRIPTIVE STATISTICS

Statistics is the collection of great masses of numerical information which is summarized and then analyzed for the purpose of making decisions; that is, the use of past information is used to predict future actions. In this chapter we will assume that the numerical data has been collected (by various processes) and will discuss distribution and measures of central tendency and variability.

FREQUENCY DISTRIBUTION

When we classify, by order, many variables into classes by size and put this information into table form, we have created a frequency distribution table.

ORDER

The grades in a mathematics class, as shown in table 3-1, are written in descending order; that is, the highest grade first, the next highest grade second, etc. We refer to this set of grades as an ordered array. If we had the grades shown in table 3-2, with the occurrences shown, we would refer to this as a frequency distribution; that is, we have shown the grades and the number of occurrences of each.

The data in table 3-2 are discrete variables and are naturally classified. By discrete variables, we mean a finite number of variables. By naturally ordered, we mean the variables are listed in increasing or decreasing order of value.

When dealing with continuous variables, we usually classify them; that is, we group them according to some class boundaries. If we were forming a frequency distribution table of weights of 100 individuals, we would determine the heaviest (231 lb) and lightest (109 lb), then divide the weights into from 10 to 20 classes. We subtracted 109 from 231 and found the difference to be 122. If we used 10 classes, each class interval would be

$$\frac{122}{10} = 12.2$$

If we used 20 classes, each class interval would be

$$\frac{122}{20} = 6.1$$

Table 3-1.—Array of values.

Grades
99
98
97
97
96
94
92
90
88
88
83
80
80
78
76
71
71
68
60

Table 3-2.—Frequency distribution.

Grade	Occurrences	Frequency
99	𝍷	1
98	𝍷𝍷	2
96	𝍷𝍷𝍷𝍷	4
92	𝍸 𝍷𝍷	7
90	𝍸	5
88	𝍸 𝍸 𝍷𝍷𝍷	13
86	𝍸 𝍸 𝍷	11
83	𝍸 𝍷𝍷	7
80	𝍸	5
78	𝍷𝍷𝍷𝍷	4
75	𝍷𝍷𝍷	3
60	𝍷	1

We may choose any number between 6.1 and 12.2, and we find it convenient to use 10 as the class interval. We know the smallest number must fall into the lowest class; therefore, we assign the lower limit of the first class as 108.5 which is one-half unit beyond the accuracy of the weights. This prevents any weight falling on a boundary. The first class interval is 108.5 (the lower boundary) plus the interval of 10 which gives 108.5 - 118.5. The next class interval is 118.5 - 128.5, etc.

Now as we determine each individual weight, we make a mark beside the proper class interval. We determine class marks by finding the midpoint of each class interval. The frequency column is the number of tally marks in the occurrence column. This is shown in table 3-3.

The class marks (x) indicate that we have assigned each weight in that class interval the weight of that class mark. The frequencies (f) indicate the number of occurrences.

HISTOGRAM

The information in table 3-3 would be easier to visualize if it were shown graphically. This is shown in figure 3-1.

The class boundaries are indicated on the horizontal axis and the frequencies are indicated on the vertical axis. If the width of the bars is considered unity, then the area of each rectangle is representative of the frequency. The total area of all the rectangles, then, represents the total frequency. Figure 3-1 is a histogram of the information in table 3-3.

POLYGON

Another method of representing the information in table 3-3 is shown in figure 3-2. This figure is developed by connecting the midpoints of the tops of adjacent rectangular bars of figure 3-1 together. These midpoints are in actual practice the class marks of the classes. The area under the curve of the polygon is the same as the area under the curve of the histogram. This may be seen by examining one of the rectangles and noting that there is the same amount of area, cut into a triangle, outside the bar as there is inside the bar. This is shown in the shaded area of figure 3-2.

Both the histogram and the polygon present a graphical representation of data which is easy to visualize. These are used to quickly compare one set of data with another set of data.

MEASURES OF CENTRAL TENDENCY

The previously mentioned frequency distribution tables dealt with the variables specifically. To summarize the tendencies of the variables we use the idea of central tendency. Several ways of describing the central tendency are by use of the arithmetic mean, the median, the mode, and the geometric and harmonic means. These are discussed in this section.

ARITHMETIC MEAN

We will use the term mean to indicate the arithmetic mean which is the commonly used idea of average.

Table 3-3.—Frequency distribution with class boundaries.

Class boundaries	Occurrences	Class marks: x	Frequencies
108.5 - 118.5	|	113.5	1
118.5 - 128.5	|||	123.5	3
128.5 - 138.5	||||	133.5	4
138.5 - 148.5	𝍸	143.5	5
148.5 - 158.5	𝍸 ||||	153.5	9
158.5 - 168.5	𝍸 𝍸 𝍸 ||	163.5	17
168.5 - 178.5	𝍸 𝍸 𝍸 𝍸	173.5	20
178.5 - 188.5	𝍸 𝍸 𝍸	183.5	15
188.5 - 198.5	𝍸 𝍸	193.5	10
198.5 - 208.5	𝍸 |||	203.5	8
208.5 - 218.5	𝍸	213.5	5
218.5 - 228.5	||	223.5	2
228.5 - 238.5	|	233.5	1

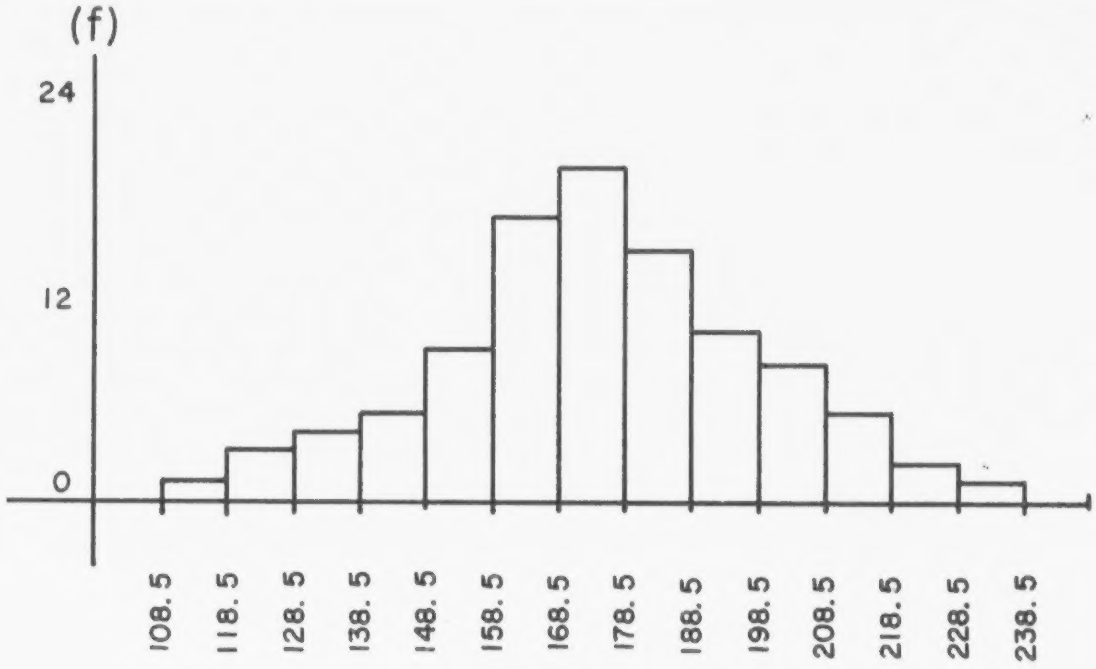


Figure 3-1.—Histogram.

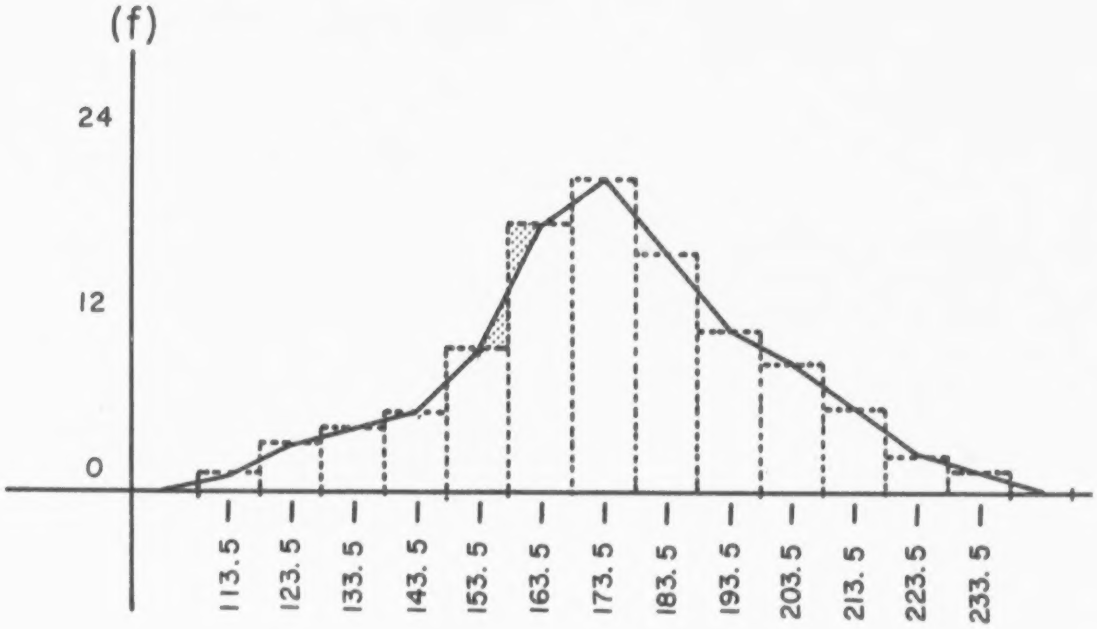


Figure 3-2.—Polygon.

The mean is defined as the sum of a group of values divided by the number of values.

If we have the test scores of 70, 68, 85, 95, 90, and 80, we find the mean by adding the scores and then dividing the sum by the number of scores we have; that is,

$$\begin{array}{r} 70 \\ 66 \\ 85 \\ 95 \\ 90 \\ \underline{80} \\ 486 \end{array}$$

and

$$486 \div 6 = 81$$

which is the mean.

If $\overline{X}$ is the mean, then

$$\overline{X} = \frac{X_1 + X_2 + X_3 + \cdots + X_n}{n}$$

or

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

Sigma (Σ) is the summation symbol and $i = 1$ to n indicates that the values of X_i from $i = 1$ to $i = n$ are added. This sum is then divided by n, the number of scores involved.

EXAMPLE: Find the mean of 78, 92, 63, 76, 83, 82, and 79.

SOLUTION: In the formula

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

the sum of the scores is 553 and n equals 7, therefore,

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

$$= \frac{553}{7}$$

$$= 79$$

In cases such as shown in table 3-4, the mean could be found by adding each grade (notice there are three 72's, six 80's, etc.) then dividing by the total number of grades. It would be far easier to multiply each grade by the number of times it occurred and then adding these products to find the sum. This sum is then divided by the total number of grades. This is shown in the formula

$$\overline{X} = \frac{\sum_{i=1}^{n} f X_i}{n}$$

where

f = frequency of each grade

and

n = total number of grades

This gives

$$\overline{X} = \frac{1(60) + 3(72) + 6(80) + 4(92) + 2(96)}{16}$$

$$= \frac{60 + 216 + 480 + 368 + 192}{16}$$

$$= 82.25$$

Table 3-4.—Sample frequency distribution.

Grades	Frequency	$f(X_i)$
60	1	60
72	3	216
80	6	480
92	4	368
96	2	192
	16	1316

Computation of Mean by Coding

Our computations to this point have dealt with a relatively small number of values. When the number of values becomes large, we may resort to the use of coding to determine the mean.

Before discussing the actual coding process, we will examine a few related type procedures. If we were to find the mean of the values 92, 87, 85, 80, 78, and 65, we could assume a mean of (80) and determine the deviation of each value from this mean as follows:

Value	Deviation
92	+ 12
87	+ 7
85	+ 5
80 assumed mean	0
78	- 2
65	- 15

We then algebraically add the deviations and find the sum to be + 7. Divide + 7 by 6, the number of values, and find the quotient to be +1.16. We then add, algebraically, the mean of the deviations (+1.16) to the assumed mean of the values (80) and find the actual mean of the values to be (80) + (+1.16) or 81.16.

If we were to find the mean of 90, 80, 70, 60, 50, and 40, we could assume the mean to be 70 and write

Value	Deviation
90	+ 20
80	+ 10
70 assumed mean	0
60	- 10
50	- 20
40	- 30

Then find the mean of the deviations to be

$$\frac{-30}{6} = -5$$

and the actual mean of the values is the algebraic sum of the assumed mean and the mean of the deviations which is

$$\begin{array}{ccccc} \text{assumed} & & \text{mean of} & & \text{actual} \\ \text{mean} & + & \text{deviations} & = & \text{mean} \\ & & & & \\ 70 & + & (-5) & = & 65 \end{array}$$

Notice in the preceding example that all deviations are multiples of ten. By dividing each by ten we would have deviations of 2, 1, 0, -1, -2, -3. Once again we shall assume 70 as the mean. We now have the following deviations and values:

Value	Deviation
90	+ 2
80	+ 1
70	0
60	- 1
50	- 2
40	- 3

The algebraic mean of the deviations is equal to

$$\frac{-3}{6} = -0.5$$

Now we must multiply -0.5 by ten to arrive at the same mean of deviations we found in the previous example. This may be done because the difference in the deviations is a constant, and this was due to the values having a constant difference.

As we have seen, the computation of the mean by use of the formula

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i f_i$$

is rather simple when the number of values and their frequencies are small. When X_i and f_i are large we may resort to the use of coding. We will use u to designate the coding variable. When we have a set of variables as shown in table 3-5, we represent the class marks by both positive and negative integers. The zero may be placed opposite any value near the middle of the distribution. We choose 163.5 as the value to correspond to u = 0.

To find the mean of the values, using the code, we find the algebraic sum of column u to be -3. The mean of the u's then, is

$$\begin{array}{r} -0.5 \\ 6\,\overline{)\,-3} \end{array}$$

Table 3-5.—Coded distribution.

Class boundaries	Frequency f	Class mark x	Code u
128.5 - 138.5	1	133.5	- 3
138.5 - 148.5	1	143.5	- 2
148.5 - 158.5	1	153.5	- 1
158.5 - 168.5	1	163.5	0
168.5 - 178.5	1	173.5	+ 1
178.5 - 188.5	1	183.5	+ 2

The differences in class marks is 10; therefore, we multiply (-0.5) by 10 and find this equal to -5. The -5 is added to the value corresponding to u equal 0. Thus,

$$163.5 + (-5) = 158.5$$

which is the mean of the class marks.

If we use x_o to designate the value corresponding to u = 0, and if we use C to indicate the class interval (difference between adjacent class marks), we may show the relationship between the x's and u's by

$$x_i = Cu_i + x_o$$

In table 3-5, the length of the class interval is 10; therefore,

$$x_i = 10u_i + x_o$$

We may verify this formula by choosing any class mark in table 3-5. Let us test

$$x_i = 143.5$$

Then

$$x_i = 10u_i + x_o$$

and

$$\begin{aligned} 143.5 &= 10\ (-2) + 163.5 \\ &= -20 + 163.5 \\ &= 143.5 \end{aligned}$$

To compute the value for $\bar{x}$ we substitute $\bar{x}$ for x_i and $\bar{u}$ for u_i in the formula

$$x_i = Cu_i + x_o$$

and find

$$\bar{x} = C\bar{u} + x_o$$

Then,

$$C = 10$$

$$\bar{u} = \frac{-3}{6} = -0.5$$

and

$$x_o = 163.5$$

therefore,

$$\begin{aligned} \bar{x} &= C\bar{u} + x_o \\ &= 10\ (-0.5) + 163.5 \\ &= -5 + 163.5 \\ &= 158.5 \end{aligned}$$

We may verify this by writing

$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$$

$$= 133.5 + 143.5 + 153.5 + 163.5 + 173.5 + 183.5$$

$$= \frac{951.0}{6}$$

$$= 158.5$$

The reason we may substitute $\bar{x}$ for x_i and $\bar{u}$ for u_i is shown as follows:

We have shown that

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i f_i \qquad (1)$$

and

$$x_i = Cu_i + x_o \qquad (2)$$

Then by substituting (2) into (1) we have

$$\bar{x} = \left(\frac{1}{n}\right) \sum_{i=1}^{n} (Cu_i + x_o) f_i$$

$$= \left(\frac{1}{n}\right) \sum_{i=1}^{n} (Cu_i f_i + x_o f_i)$$

$$= \left(\frac{1}{n}\right) \sum_{i=1}^{n} Cu_i f_i + \left(\frac{1}{n}\right) \sum_{i=1}^{n} x_o f_i$$

$$= C \left(\frac{1}{n}\right) \sum_{i=1}^{n} u_i f_i + x_o \left(\frac{1}{n}\right) \sum_{i=1}^{n} f_i$$

Now,

$$\bar{u} = \left(\frac{1}{n}\right) \sum_{i=1}^{n} u_i f_i$$

and

$$\left(\frac{1}{n}\right) \sum_{i=1}^{n} f_i = 1, \text{ where } n = \sum_{i=1}^{n} f_i$$

therefore,

$$C \left(\frac{1}{n}\right) \sum_{i=1}^{n} u_i f_i$$

$$= C\bar{u}$$

and

$$x_o \left(\frac{1}{n}\right) \sum_{i=1}^{n} f_i = x_o$$

then

$$\bar{x} = C \left(\frac{1}{n}\right) \sum_{i=1}^{n} u_i f_i + x_o \left(\frac{1}{n}\right) \sum_{i=1}^{n} f_i$$

$$= C\bar{u} + x_o$$

The previous example which used table 3-5 dealt with values which all had a frequency of one. To compute the mean of the values shown in table 3-6 will involve varied frequencies and is done as follows:

We use the formula

$$\bar{x} = C\bar{u} + x_o \qquad (3)$$

and by inspection of table 3-6 find that

$$C = 10$$

and

$$x_o = 163.5$$

The next step is to determine u; that is,

$$\bar{u} = \left(\frac{1}{n}\right) \sum_{i=1}^{n} u_i f_i$$

where

$$n = 42$$

and

$$\sum_{i=1}^{n} u_i f_i = +9$$

Then,

$$\bar{u} = \frac{1}{42} \left(+ \frac{9}{1}\right)$$

$$= \frac{9}{42}$$

Table 3-6.—Coded frequency distribution.

Class boundaries	Frequency f	Class marks x	Code u	(Code) (Freq.) uf
128.5 - 138.5	2	133.5	- 3	- 6
138.5 - 148.5	4	143.5	- 2	- 8
148.5 - 158.5	7	153.5	- 1	- 7
158.5 - 168.5	11	163.5	0	0
168.5 - 178.5	9	173.5	+ 1	+ 9
178.5 - 188.5	6	183.5	+ 2	+ 12
188.5 - 198.5	3	193.5	+ 3	+ 9
	42			+ 9

Substituting into equation (3), find that

$$\bar{x} = 10\left(\frac{9}{42}\right) + 163.5$$

$$= \frac{90}{42} + 163.5$$

$$= 2.14 + 163.5$$

$$= 165.64$$

To show the usefulness of coding, we will now compute (the long way) the mean of the class marks of table 3-6 by use of the formula

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i f_i$$

$$= \frac{1}{42}\sum_{i=1}^{n} x_i f_i$$

and the products of the $x_i f_i$'s are

$133.5 \times 2 = 267.0$

$143.5 \times 4 = 574.0$

$153.5 \times 7 = 1074.5$

$163.5 \times 11 = 1798.5$

$173.5 \times 9 = 1561.5$

$183.5 \times 6 = 1101.0$

$193.5 \times 3 = 580.5$

Therefore,

$$\frac{1}{42}\sum_{i=1}^{n} x_i f_i$$

$$= \frac{1}{42}(6957.0)$$

$$= \frac{6957.0}{42}$$

$$= 165.64$$

Notice that $\bar{x}$ was the same in each case, but in the first case far less computation was required.

PROBLEMS:

1. Find $\bar{x}$ in table 3-7 by completing the indicated columns and using coding, and check your answer by using the formula

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i f_i$$

2. Compute, by coding, $\bar{x}$ in table 3-3.

ANSWERS:

1. $\bar{x} = 182.43$

2. $\bar{x} = 174.3$

Table 3-7.—Practice problem.

Class boundaries	Frequency f	Class marks x	Code u	(Code) (Freq.) uf
138.2 - 148.2	4	143.2		
148.2 - 158.2	6	153.2		
158.2 - 168.2	13	163.2		
168.2 - 178.2	15			
178.2 - 188.2	17			
188.2 - 198.2	14			
198.2 - 208.2	11			
208.2 - 218.2	8			
218.2 - 228.2	2			
	90			

MEDIAN

In an ordered array of values, the value which has as many values above it as below it is called the median. In some cases the median may be a point rather than a value. This occurs when there is an even number of values and will be explained by a following example.

When we have a very large or small value, as compared to the other values in the array, the median is generally superior to the mean as a measure of central tendency. This is because the large or small value will cause the mean to move away from the major grouping of the values.

EXAMPLE: Compare the mean and median of the values 2, 3, 3, 5, 6, 9, 132.

SOLUTION: We find the mean to be

$$\frac{161}{7} = 23$$

The median is the middle number in the ordered array

132
9
6
5
3
3
2

which is the number 5. Notice that more of the values cluster above and below the 5 than about the mean of 23.

Also, the median may be used in cases where items are arranged according to merit rather than value. For example, workers may be rated by their ability; then, the median of the abilities of the workers is the rating of the middle worker in the array.

Generally, then, the median is a measure of position rather than a measure of value.

EXAMPLE: Find the median of the following values: 7, 6, 5, 3, 2, 2, 1.

SOLUTION: Arrange the values in an ordered array as

7
6
5
3
2
2
1

Then, by inspection, find the number which has as many values above it as below it: 3.

EXAMPLE: Find the median of the values 9, 9, 7, 6, 5, 4.

SOLUTION: The values in ordered array form are

9
9
7
6
5
4

We find no middle value; therefore, the median is the mean of the two middle values; that is,

$$\overline{X} = \frac{7 + 6}{2}$$

$$= 6.5$$

which is the median for the set of values given.

Notice that $\overline{X}$ in this case is the mean of only the two values 6 and 7 and not the entire array. We will designate the median by Md.

EXAMPLE: Find and compare the median of each of the following arrays; that is, A and B:

A	B
9	50
8	48
7	10
6	6
5	3
4	2
3	1

SOLUTION: The median of both A and B is 6. In this case the median should not be used to compare A and B because of the wide range of B values and the close grouping of A values. In this case the means would give a better comparison. In some cases the median will give a more realistic meaning to a set of values.

EXAMPLE: In a small organization the salaries of the 5 employees are

Employee	Salary
A	\$ 7600
B	\$ 4900
C	\$ 4700
D	\$ 4500
E	\$ 4300

The median of the salaries is \$ 4700. The mean of the salaries is

$$\overline{X} = \frac{7600 + 4900 + 4700 + 4500 + 4300}{5}$$

$$= \frac{26000}{5}$$

$$= \$\ 5200$$

Notice that with

$$Md = \$\ 4700$$

and

$$\overline{X} = \$\ 5200$$

the median is more representative than the mean because four of the five employees have salaries less than the mean.

MODE

In a distribution the value which occurs most often is called the mode. When two or more values occur most often, rather than just one value, there will be more than one mode.

EXAMPLE: Find the mode of the values 7, 9, 11, 7, 8, 6, 6, 7, 5.

SOLUTION: By inspection, the value which occurs most often is 7; therefore, the mode is 7. It is indicated by writing

$$Mo = 7$$

EXAMPLE: Find the mode of the values 18, 20, 17, 17, 16, 16, 15, 16, 17, 20.

SOLUTION: By inspection, the values which occur most often are 16 and 17; therefore,

$$Mo = 16 \text{ and } 17$$

In this case there are two modes.

RANGE

The range of a set of values or of an array is defined as the difference between the largest and smallest values. The range in the preceding example is the high value (20) minus the low value (15) which is

$$20 - 15 = 5 = r \text{ (range)}$$

PROBLEMS: Find the mean, median, mode, and range in the following:

1. 17, 19, 31, 21, 34, 6, 8, 9, 17

2. 100, 60, 80, 80, 60, 60, 70, 50

3. 7.2, 3.7, 6.2, 10.3, 11.9

ANSWERS:

1. $\overline{X} = 18$

   $Md = 17$

   $Mo = 17$

   $r = 28$

2. $\overline{X} = 70$

$Md = 65$

$Mo = 60$

$r = 50$

3. $\overline{X} = 7.86$

$Md = 7.2$

Mo = none

$r = 8.2$

GEOMETRIC MEAN

The geometric mean is sometimes used to average a set of percentages or other ratios. The geometric mean is found in the same manner as the arithmetic mean except that logarithms are used. The geometric mean is also useful when the variables have the characteristics of a geometric progression or sequence. The formula for the geometric mean is

$$G = \sqrt[n]{X_1 \cdot X_2 \cdot X_3 \cdots X_n}$$

This formula may be changed as follows, by taking logarithms on both sides:

then

$$\log G = \log \sqrt[n]{X_1 \cdot X_2 \cdot X_3 \cdots X_n}$$

$$= \log (X_1 \cdot X_2 \cdot X_3 \cdots X_n)^{\frac{1}{n}}$$

$$= \left(\frac{1}{n}\right) \log (X_1 \cdot X_2 \cdot X_3 \cdots X_n)$$

$$= \frac{\log (X_1 \cdot X_2 \cdot X_3 \cdots X_n)}{n}$$

$$= \frac{\log X_1 + \log X_2 + \log X_3 + \cdots + \log X_n}{n}$$

$$\log G = \frac{\sum_{i=1}^{n} \log x_i}{n}$$

By definition, the geometric mean of X is the antilogarithm of the arithmetic mean of log X.

EXAMPLE: Find the arithmetic mean and the geometric mean of the price:earning ratios for the stocks as shown in table 3-8.

SOLUTION: Find the logarithms of X and write the column of log X and find totals.

Table 3-8.—Geometric mean.

Stock	Price earning ratio (x)	Log of price:earning ratio (log x)
I	18.2	1.2601
II	17.3	1.2380
III	16.8	1.2253
IV	14.5	1.1614
V	31.2	1.4942
	98.0	6.3790

The arithmetic mean is

$$\overline{X} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$= \frac{98}{5}$$

$$= 19.6$$

The geometric mean is

$$\log G = \frac{\sum_{i=1}^{n} \log x_i}{n}$$

$$= \frac{6.3790}{5}$$

$$= 1.2758$$

Then,

G = antilogarithm of log G

= antilog (1.2758)

$= 18.8^{+}$

PROBLEM: Find the arithmetic and geometric mean of the following:

Item	X	log X
I	32.6	1.5132
II	17.2	1.2355
III	9.6	0.9823
IV	21.7	1.3365
V	33.1	1.5198
VI	15.8	1.1987

ANSWER:

arithmetic mean is 21.66

geometric mean is 19.84

HARMONIC MEAN

In certain cases the harmonic mean serves a useful purpose. Although this mean is generally not found in statistics, it is a method of describing a set of numbers and will be explained.

When averages are desired where equal times are involved, the arithmetic mean of speeds is used and when equal distances are given the harmonic mean is useful.

EXAMPLE: An automobile travels for 3 hours at a rate of 60 miles per hour, then travels for 3 hours at a rate of 70 miles per hour. What is the average speed of the automobile?

SOLUTION: Equal times are involved; therefore, the arithmetic mean of the automobile speed is

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

$$\overline{X} = \frac{60 + 70}{2}$$

$$= \frac{130}{2}$$

$$= 65$$

EXAMPLE: An automobile travels 200 miles at a rate of 60 miles per hour and the next 200 miles at a rate of 70 miles per hour. What is the average speed of the automobile?

SOLUTION: Equal distances are involved, resulting in unequal times for the two rates. Therefore, the harmonic mean is more accurate than the arithmetic mean. It is found as follows:

$$H = \frac{n}{\frac{1}{X_1} + \frac{1}{X_2} + \cdots + \frac{1}{X_n}}$$

$$= \frac{n}{\sum_{i=1}^{n} \frac{1}{X_i}}$$

$$= \frac{2}{\frac{1}{60} + \frac{1}{70}}$$

$$= \frac{2}{\frac{130}{4200}}$$

$$= 2\left(\frac{420}{13}\right)$$

$$= 64.6$$

The difference in the averages is explained by the fact that the automobile in the first example traveled a total distance of 390 miles in 6 hours or at an average speed of 65 miles per hour. The automobile in the second example traveled 400 miles in 3.33 hours plus 2.86 hours or 6.19 hours which is an average speed of 64.6 miles per hour.

The fallacy of using "averaging" (arithmetic mean) when times are unequal may be demonstrated even more dramatically by finding the "average" speed of an automobile which travels 595 miles at a speed of 60 miles per hour and travels the final 5 miles of a 600-mile trip at 20 miles per hour.

PROBLEM: An automobile travels 100 miles at a speed of 50 miles per hour, 100 miles at a speed of 45 miles per hour and 100 miles at a speed of 70 miles per hour. What is the average speed of the automobile?

ANSWER: 53.09 miles per hour.

MEASURES OF VARIABILITY

To this point we have discussed averages or means of sets of values. While the mean is a useful tool in describing a characteristic of a set of values, it does not indicate how the values are dispersed about the mean. That is, the values 20, 50, and 80 have the same mean as 45, 50, and 55 although in the first case the dispersion and range is much greater. In describing a set of values we need to know not only the mean but also how the values are dispersed about the mean.

Generally, when the dispersion is small, the average is a reliable description of the values; and if the dispersion is great, the average is not typical of the values, unless the number of values is very large.

MEAN DEVIATION

The mean deviation is defined as the arithmetic mean of the absolute values of the deviations minus the mean. In the set of values 45, 50, and 55, the mean deviation, given by the formula

$$\text{M. D.} = \frac{1}{n}\sum_{i=1}^{n} |X_i - \overline{X}|$$

where the mean $(\overline{X})$ is 50, is shown as

values X	$\lvert X_i - \overline{X}\rvert$	= absolute value of deviations
55	$\lvert 55-50\rvert$	= 5
50	$\lvert 50-50\rvert$	= 0
45	$\lvert 45-50\rvert$	= 5

and the mean of the absolute values of the deviations is

$$\frac{5+0+5}{3}$$

$$= \frac{10}{3}$$

$$= 3.33$$

Notice that the absolute value of the deviations is used because the mean of the deviations would be zero; that is,

$$X_i - \overline{X} = \text{deviation}$$

$$55-50 = 5$$

$$50-50 = 0$$

$$45-50 = -5$$

and the mean would be

$$\frac{5+0-5}{3}$$

$$= \frac{0}{3}$$

$$= 0$$

The mean deviation of the values 20, 50, and 80 is

$$\text{M. D.} = \frac{1}{n}\sum_{i=1}^{n} |X_i - \overline{X}|$$

$$= \frac{1}{3}(30 + 0 + 30)$$

$$= \frac{60}{3}$$

$$= 20$$

EXAMPLE: Find the mean deviation of the values 72, 60, 85, 90, 63, 80, 90, 93, and 87.

SOLUTION: Make an array with columns for X and $|X_i - \overline{X}|$ as follows:

X	$\lvert X_i - \overline{X}\rvert$
93	
90	
90	
87	
85	
80	
72	
63	
60	

Determine the mean as

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$= \frac{1}{9}(720)$$

$$= 80$$

then complete the column for $|X_i - \overline{X}|$ as

X	$\lvert X_i - \overline{X}\rvert$
93	13
90	10
90	10
87	7
85	5
80	0
72	8
63	17
60	20

Now,

$$\text{M. D.} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})$$

$$= \frac{1}{9} (13 + 10 + 10 + 7+5+0 + 8 + 17 + 20)$$

$$= \frac{90}{9}$$

$$= 10$$

PROBLEMS: Find the mean deviation for each set of values given.

1. 7, 9, 11, 20, 15
2. 2, 2, 3, 4, 5

ANSWERS:

1. 4.08
2. 1.04

To this point in our discussion of mean deviation we have dealt with arrays of values. If we desire to find the mean deviation of a frequency distribution, we need only modify the formula for mean deviation; that is,

$$\text{M. D.} = \frac{1}{n} \sum_{i=1}^{n} |X_i - \overline{X}|$$

is written to include frequency (f_i) as follows:

$$\text{M. D.} = \frac{1}{n} \sum_{i=1}^{n} |X_i - \overline{X}|\, f_i$$

EXAMPLE: Find the mean deviation of the following values:

16, 13, 15, 15, 13, 17, 13, 18, 20, 17, 12

SOLUTION: Write the frequency distribution as follows:

X	f	$\lvert X_i - \overline{X}\rvert$	$\lvert X_i - \overline{X}\rvert f_i$
20	1	4.6	4.6
18	1	2.6	2.6
17	2	1.6	3.2
16	1	0.6	0.6
15	2	0.4	0.8
13	3	2.4	7.2
12	1	3.4	3.4

where $\overline{X} = 15.4$ and

$$\text{M. D.} = \frac{1}{n} \sum_{i=1}^{n} |X_i - \overline{X}|\, f_i$$

$$= \frac{1}{11} [1(4.6) + 1(2.6) + 2(1.6) + 1(0.6) + 2(0.4) + 3(2.4) + 1(3.4)]$$

$$= \frac{1}{11} (4.6 + 2.6 + 3.2 + 0.6 + 0.8 + 7.2 + 3.4)$$

$$= 2.04$$

PROBLEM: Find the mean deviation of the grades in table 3-4.

ANSWER: 8.31

STANDARD DEVIATION

While the mean deviation is a useful tool in statistics, the standard deviation is the most important measure of variability. The standard deviation is the square root of the mean of the squares of the deviation minus the mean; that is,

$$\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2}$$

The symbol σ (lower case sigma) designates the standard deviation. Notice that instead of using the absolute value of $X_i - \overline{X}$ as in the computation of the mean deviation, we square $X_i - \overline{X}$ and then find the square root of the sum of $(X_i - \overline{X})^2$ divided by n.

EXAMPLE: Find the standard deviation of the values 60, 70, 75, 65, 70, 80.

SOLUTION: Make a table as follows:

X	$X_i - \overline{X}$	$(X_i - \overline{X})^2$
80	+ 10	100
75	+ 5	25
70	0	0
70	0	0
65	- 5	25
60	- 10	100
		250

Then,

$$\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2}$$

$$= \sqrt{\frac{1}{6} (250)}$$

$$= \sqrt{41.7}$$

$$= 6.45$$

This indicates that the standard deviation of the values from the mean of 70 is 6.45.

EXAMPLE: Find the standard deviation of the values 2, 2, 3, 4, 5.

SOLUTION: Write

X	$X_i - \overline{X}$	$(X_i - \overline{X})^2$
2	-1.2	1.44
2	-1.2	1.44
3	-0.2	0.04
4	+0.8	0.64
5	+1.8	3.24
		6.80

Therefore,

$$\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2}$$

$$= \sqrt{\frac{1}{5} (6.80)}$$

$$= \sqrt{1.36}$$

$$= 1.166$$

In the two previous examples we used n as the divisor, but in many cases, especially where n is small, the formula is modified by the use of n - 1 in place of n; that is,

$$\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2}$$

is the standard deviation for a large population and

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2}$$

is the formula for standard deviation when the population is small. In some cases, s is called the sample standard deviation. The latter is commonly used in statistics. In the previous example the value for s is

$$s = \sqrt{\frac{1}{4} (6.80)}$$

$$= \sqrt{1.70}$$

$$= 1.3$$

and gives a better estimate of the standard deviation of the population from which the sample was taken.

We have shown examples where the frequency of occurrence of each value was considered individually. To use the formula for standard deviation with a frequency distribution, we need only include f_i; that is,

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2 f_i}$$

EXAMPLE: Find the standard deviation of the values 80, 75, 75, 70, 65, 65, 65, 60.

SOLUTION: Write

X	f
80	1
75	2
70	1
65	3
60	1

Then,

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i f_i$$

$$= \frac{1}{8}\,[1(80) + 2(75) + 1(70) + 3(65) + 1(60)]$$

$$= \frac{1}{8}\,(80 + 150 + 70 + 195 + 60)$$

$$= 69.37$$

Now write the following tabulation:

X	f	$X_i - \bar{X}$	$(X_i - \bar{X})^2$	$(X_i - \bar{X})^2 f_i$
80	1	10.63	112.99	112.99
75	2	5.63	31.69	63.38
70	1	.63	.39	.39
65	3	-4.37	19.09	57.27
60	1	-9.37	87.79	87.79
				321.82

therefore,

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 f_i}$$

$$= \sqrt{\frac{1}{7}\,(321.82)}$$

$$= \sqrt{45.97}$$

$$= 6.78$$

In order to simplify our calculations, we resort to the use of coding as we did with the mean. We know that

$$X_i = Cu_i + X_o$$

and

$$\bar{X} = C\bar{u} + X_o$$

and by subtracting the second equation from the first equation we have

$$\begin{array}{rrcl} & X_i & = & Cu_i + X_o \\ (-) & \bar{X} & = & C\bar{u} + X_o \\ \hline & X_i - \bar{X} & = & Cu_i + X_o - (C\bar{u} + X_o) \\ & & = & Cu_i + X_o - C\bar{u} - X_o \\ & & = & Cu_i - C\bar{u} \\ & & = & C(u_i - \bar{u}) \end{array}$$

Substitute

$$X_i - \bar{X} = C(u_i - \bar{u})$$

in the formula for standard deviation

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 f_i}$$

$$= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}[C(u_i - \bar{u})]^2 f_i}$$

$$= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}C^2(u_i - \bar{u})^2 f_i}$$

$$= C\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2 f_i}$$

In the previous example, because the difference between values is a constant which is 5, we may write

X	f_i	u_i	$u_i f_i$	$u_i - \bar{u}$	$(u_i - \bar{u})^2$	$(u_i - \bar{u})^2 f_i$
80	1	+2	+2	2.125	4.51	4.51
75	2	+1	+2	1.125	1.26	2.52
70	1	0	0	0.125	0.0156	0.0156
65	3	-1	-3	-0.875	0.76	2.28
60	1	-2	-2	-1.875	3.51	3.51
	8					12.84

where

$$\bar{u} = \frac{1}{n} \sum_{i=1}^{n} u_i f_i$$

$$= \frac{1}{8} (-1)$$

$$= -\frac{1}{8}$$

$$= -0.125$$

Then,

$$s = C \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2 f_i}$$

$$= C \sqrt{\frac{1}{n-1} (12.84)}$$

$$= 5 \sqrt{\frac{12.84}{7}}$$

$$= 5 \sqrt{1.834}$$

$$= 5 (1.354)$$

$$= 6.77$$

A simpler method for calculating the standard deviation is by changing the formula

$$s = C \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2 f_i}$$

by the following algebraic manipulations and, omitting the limits to simplify calculations, we have

$$\Sigma (u_i - \bar{u})^2 f_i$$

$$= \Sigma u_i^2 f_i - 2 \Sigma u_i \bar{u} f_i + \Sigma \bar{u}^2 f_i$$

$$= \Sigma u_i^2 f_i - 2 \bar{u} \Sigma u_i f_i + \bar{u}^2 \Sigma f_i$$

$$= \Sigma u_i^2 f_i - 2 \left(\frac{1}{n}\right) (\Sigma u_i f_i)^2 + \left(\frac{\Sigma u_i f_i}{n}\right)^2 \Sigma f_i$$

$$= \Sigma u_i^2 f_i - \left(\frac{2}{n}\right) (\Sigma u_i f_i)^2 + (\Sigma u_i f_i)^2 \frac{\Sigma f_i}{n^2}$$

$$= \Sigma u_i^2 f_i - \left(\frac{2}{n}\right) (\Sigma u_i f_i)^2 + (\Sigma u_i f_i)^2 \left(\frac{1}{n}\right)$$

$$= \Sigma u_i^2 f_i - \left(\frac{1}{n}\right) (\Sigma u_i f_i)^2$$

therefore,

$$s = C \sqrt{\frac{1}{n-1} \left[\sum_{i=1}^{n} u_i^2 f_i - \frac{1}{n} \left(\sum_{i=1}^{n} u_i f_i \right)^2 \right]}$$

The previous example is solved by writing

X	f	u	uf	u^2 f
80	1	+2	+2	4
75	2	+1	+2	2
70	1	0	0	0
65	3	-1	-3	3
60	1	-2	-2	4
			-1	13

and

$$s = C \sqrt{\frac{1}{n-1} \left[\Sigma u_i^2 f_i - \frac{1}{n} (\Sigma u_i f_i)^2 \right]}$$

$$= 5 \sqrt{\frac{1}{7} \left[13 - \frac{1}{8} (1) \right]}$$

$$= 5 \sqrt{\frac{1}{7} \left(12 \frac{7}{8} \right)}$$

$$s = 5\sqrt{\frac{1}{7}\;\frac{(103)}{8}}$$

$$= 5\sqrt{\frac{103}{56}}$$

$$= 5\sqrt{1.84}$$

$$= 5\ (1.36)$$

$$= 6.80$$

EXAMPLE: Find the standard deviation of the values

X	f	u	uf	u^2 f
80	3	+2	6	12
70	4	+1	4	4
60	7	0	0	0
50	6	-1	-6	6
40	4	-2	-8	16
30	2	-3	-6	18
	26		-10	56

SOLUTION: Write

$$s = C\sqrt{\frac{1}{n-1}\left[\Sigma\, u_i^2 f_i - \frac{1}{n}\left(\Sigma\, u_i f_i\right)^2\right]}$$

$$= 10\sqrt{\frac{1}{25}\left[56 - \frac{1}{26}\ (100)\right]}$$

$$= 10\sqrt{\frac{1}{25}\left(56 - \frac{100}{26}\right)}$$

$$= 10\sqrt{1.80}$$

$$= 10\ (1.34)$$

$$= 13.4$$

PROBLEM: Find the standard deviation, by coding, of

X	f
86	1
81	3
76	11
71	13
66	9
61	4
56	2

ANSWER: Approximately 6.6.

When calculating the standard deviation of ungrouped or raw values, we may use, instead of

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

the formula

$$s = \sqrt{\frac{\Sigma\, X^2 - \frac{(\Sigma\, X)^2}{n}}{n-1}}$$

where X is the symbol representing the original values.

EXAMPLE: Find the standard deviation of the values 80, 75, 75, 70, 65, 65, 65, and 60.

SOLUTION: Write

	X	X^2
	80	6400
	75	5625
	75	5625
	70	4900
	65	4225
	65	4225
	65	4225
	60	3600
Totals	555	38825

then,

$$s = \sqrt{\frac{\Sigma\, X^2 - \frac{(\Sigma\, X)^2}{n}}{n-1}}$$

$$= \sqrt{\frac{38825 - \frac{(555)^2}{8}}{7}}$$

$$= \sqrt{\frac{38825 - 38503}{7}}$$

$$= \sqrt{\frac{322}{7}}$$

$$= \sqrt{46}$$

$$= 6.78$$

which agrees with a previous problem in which we used the formula

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 f_i}$$

Notice that f_i is included in this formula. We could have grouped our values and used the formula

$$s = \sqrt{\frac{\Sigma X^2 f_i - \frac{(\Sigma X f_i)^2}{n}}{n-1}}$$

which is the formula for grouped values.

EXAMPLE: Compare the standard deviation of the values 82, 80, 80, 78, 77, 63, and 60 found by both

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 f_i}$$

and

$$s = \sqrt{\frac{\Sigma X^2 f_i - \frac{(\Sigma X f_i)^2}{n}}{n-1}}$$

SOLUTION: Write

X	f	Xf	$X_i - \bar{X}$	$(X_i - \bar{X})^2$	$(X_i - \bar{X})^2 f$	$X^2 f_i$
82	1	82	+7	49	49	6724
80	2	160	+5	25	50	12800
78	1	78	+3	9	9	6084
77	1	77	+2	4	4	5929
66	1	66	-9	81	81	4356
62	1	62	-13	169	169	3844
Totals		525			362	39737

where

$$\bar{X} = \frac{1}{n} \Sigma X_i f_i$$

$$= \frac{1}{7} (525)$$

$$= 75$$

Then,

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 f_i}$$

$$= \sqrt{\frac{1}{6} (362)}$$

$$= \sqrt{\frac{362}{6}}$$

$$= \sqrt{60.3}$$

$$= 7.76$$

and

$$s = \sqrt{\frac{\Sigma X^2 f_i - \frac{(\Sigma X f_i)^2}{n}}{n-1}}$$

$$= \sqrt{\frac{39717 - \frac{275625}{7}}{6}}$$

$$= \sqrt{\frac{362}{6}}$$

$$= \sqrt{60.3}$$

$$= 7.76$$

CHAPTER 4

STATISTICAL INFERENCE

Statistical inference is the process of making "inferences," or deductions, concerning large numbers.

Statistical processes are based on studies of large amounts of data. However, it is virtually impossible to examine each person or object in a large group (population). Therefore, the common practice is to select a representative sample group of manageable size for detailed study.

This may be seen in the problem of determining the average height of all 15 year old boys in the United States. It would be impossible to measure each boy; therefore, a representative group is taken from the population and measured, then an inference is made to the population.

Prior to the discussion of sampling we will review combinations, permutations, and probability distributions (which were discussed in Mathematics, Vol. 2, NavPers 10071-B) and the interpretation of standard deviation.

REVIEW

This section is the brief review of combinations, permutations, and probability. These subjects will be discussed in order that they may be related to distributions.

COMBINATIONS

A combination is defined as a possible selection of a certain number of objects taken from a group with no regard given to order. For instance, if we choose two letters from A, B, and C, we could write the letters as

AB, AC, and BC

The order in which we wrote the letters is of no concern; that is, AB could be written BA but we would still have only one combination of the letters A and B.

The general formula for possible combinations of n objects taken r at a time is

$$_nC_r = \frac{n!}{r!(n-r)!}$$

EXAMPLE: If we have available seven men and need a working party of four men, how many different groups may we possibly select?

SOLUTION: Write

$$_nC_r = \frac{n!}{r!(n-r)!}$$

where

$$n = 7$$

and

$$r = 4$$

Then,

$$_nC_r = \frac{7!}{4!(7-4)!}$$

$$= \frac{7!}{4!\,3!}$$

$$= \frac{5 \cdot 6 \cdot 7}{3 \cdot 2 \cdot 1}$$

$$= 35$$

Principle of Choice

If a selection can be made in n ways, and after this selection is made, a second selection can be made in n_2 ways, and so forth for r ways, then the r selections can be made together in

$$n_1 \cdot n_2 \cdot n_3 \cdot \cdots n_r \text{ ways}$$

EXAMPLE: In how many ways can a coach choose first a football team and then a basketball team if twenty boys go out for either team?

SOLUTION: The coach first may choose a football team. Write

$$
\begin{aligned}
{}_nC_r &= \frac{20!}{11!(20-11)!} \\
&= \frac{20!}{11!\,9!} \\
&= \frac{12\cdot 13\cdot 14\cdot 15\cdot 16\cdot 17\cdot 18\cdot 19\cdot 20}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} \\
&= 167{,}960
\end{aligned}
$$

The coach then chooses a basketball team from the remaining nine boys. Write

$$
\begin{aligned}
{}_nC_r &= \frac{9!}{5!(9-5)!} \\
&= \frac{9!}{5!\,4!} \\
&= \frac{6\cdot 7\cdot 8\cdot 9}{4\cdot 3\cdot 2} \\
&= 126
\end{aligned}
$$

Then, by the principle of choice, the coach may choose the two teams together in

$$(167{,}960)\,(126) = 21{,}162{,}960 \text{ ways}$$

PERMUTATIONS

Permutations are similar to combinations but extend the requirements of combinations by considering order. If we choose the letters A and B, we have only one combination; that is, AB—but we have two permutations. The two permutations are AB and BA where order is considered.

The general formula for possible permutations of n objects taken r at a time is

$${}_nP_r = \frac{n!}{(n-r)!}$$

EXAMPLE: If six persons are to fill three different positions in a company, in how many ways is it possible to fill the positions?

SOLUTION: Since any person may fill any position, we have a permutation of

$$
\begin{aligned}
{}_6P_3 &= \frac{n!}{(n-r)!} \\
&= \frac{6!}{3!} \\
&= 4\cdot 5\cdot 6 \\
&= 120
\end{aligned}
$$

Principle of Choice

The principle of choice holds for permutations as well as combinations.

EXAMPLE: Two positions are to be filled from a group of seven people. One position requires two people and the other requires three. In how many ways may the positions be filled?

SOLUTION: The first position may be filled by

$$
\begin{aligned}
{}_nP_r &= \frac{7!}{(7-2)!} \\
&= \frac{7!}{5!} \\
&= 6\cdot 7 \\
&= 42
\end{aligned}
$$

and the second position may be filled using the remaining five people; that is,

$$
\begin{aligned}
{}_nP_r &= \frac{5!}{(5-3)!} \\
&= \frac{5!}{2!} \\
&= 60
\end{aligned}
$$

Therefore, both positions may be filled in

$$(42)\,(60) = 2520$$

PROBABILITY

This section covers a review of probability in a somewhat new or different approach to the subject.

For an event that will result in any of n equally likely ways, with s indicating success and f indicating failure, the probability of success is

$$p = \frac{s}{s + f}$$

where

$$s + f = n$$

EXAMPLE: What is the probability that a die will land with a six showing?

SOLUTION: There is only one successful way the die can land and there are five ways of failure, therefore,

$$s = 1$$

$$f = 5$$

and

$$p = \frac{s}{s + f}$$

$$= \frac{1}{5 + 1}$$

$$= \frac{1}{6}$$

If we assign the letter q to be the probability of failure and p the probability of success then,

$$p = \frac{s}{s + f}$$

and

$$q = \frac{f}{s + f}$$

and

$$p + q = \frac{s}{s + f} + \frac{f}{s + f} = 1$$

In the case of the die in the preceding example, the probability of failure of the six showing is

$$f = 5$$

and

$$q = \frac{f}{s + f}$$

$$= \frac{5}{1 + 5}$$

$$= \frac{5}{6}$$

and

$$p + q = \frac{1}{6} + \frac{5}{6}$$

$$= 1$$

Although the previous example is not a practical one, we will use this approach in our discussion of probability for the sake of understanding. The same rules we will discuss may be applied to practical situations, especially where the relative frequency is determined on the basis of adequate statistical samples. In these cases relative frequency is a close approximation to probability. Relative frequency is defined as the number of successful events divided by the total number of events. Relative frequency is empirical in nature; that is, it is deduced from previous occurrences.

When a coin is tossed three times, the probability that it falls heads exactly one time is shown in the outcomes as

$$\text{T T H, T H T, H T T} \qquad (1)$$

and the other outcomes are

$$\text{T T T, T H H, H T H, T H H, H H H} \qquad (2)$$

where group (1) are the favorable outcomes and group (2) are the unfavorable outcomes.

The probability that event A occurs is the ratio of the number of times A occurs to the total possible outcomes. If we let P{A} denote the probability that event A will occur; let n (A) denote the number of outcomes which produce A; and let N denote the total number of outcomes, we may show this as

$$P\{A\} = \frac{n(A)}{N}$$

$$= \frac{3}{8}$$

which is really the favorable outcomes divided by the total number of trials.

In our first example, that of tossing a die, the probability of a six showing face up, if we let B be the event of the six showing, is given by

$$P\{B\} = \frac{n(B)}{N}$$

$$= \frac{1}{6}$$

Mutually Exclusive Events

Two or more events are called mutually exclusive if the occurrence of any one of them excludes the occurrence of the others. If two events are A and B, then

$$P\{A + B\} = \frac{n(A) + n(B)}{N}$$

$$= \frac{n(A)}{N} + \frac{n(B)}{N}$$

and

$$P\{A\} = \frac{n(A)}{N}$$

and

$$P\{B\} = \frac{n(B)}{N}$$

therefore

$$P\{A + B\} = P\{A\} + P\{B\}$$

This is called the addition rule.

EXAMPLE: What is the probability of a five or a six showing face up if a die is tossed?

SOLUTION: Let A be the five and B be the six. Then,

$$P\{A\} = \frac{1}{6}$$

and

$$P\{B\} = \frac{1}{6}$$

therefore,

$$P\{A + B\} = P\{A\} + P\{B\}$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{1}{3}$$

Dependent Events

Two or more events are said to be dependent if the occurrence or nonoccurrence of one of the events affects the probabilities of occurrence of any of the others. If two events are A and B then,

$$P\{AB\} = P\{A\}\, P\{B|A\}$$

which indicates that the probability of two events occurring is equal to the probability of the first (A) times the probability of the second (B) when it is known that A has occurred. This is called the multiplication rule and $P\{B|A\}$ is referred to as conditional probability.

EXAMPLE: What is the probability of drawing, in two successive draws (one marble at a time), two black marbles if a box contains three white and two black marbles?

SOLUTION: Let the draws be A and B. Then

$$P\{AB\} = P\{A\}\, P\{B|A\}$$

and

$$P\{A\} = \frac{2}{5}$$

and

$$P\{B|A\} = \frac{1}{4}$$

therefore

$$P\{AB\} = \frac{2}{5} \cdot \frac{1}{4}$$

$$= \frac{1}{10}$$

If two events A and B do not affect each other, then A is said to be independent of B and we write

$$P\{AB\} = P\{A\}\,P\{B\}$$

where A and B are independent.

EXAMPLE: What is the probability that from a box containing three white and two black marbles we draw a white marble, replace it, and then draw a black marble?

SOLUTION: Let A and B be the events respectively, then

$$P\{AB\} = P\{A\}\,P\{B\}$$

because one event does not affect the other event. Then

$$P\{A\} = \frac{3}{5}$$

and

$$P\{B\} = \frac{2}{5}$$

therefore

$$P\{AB\} = \frac{3}{5} \cdot \frac{2}{5}$$

$$= \frac{6}{25}$$

DISTRIBUTIONS

In order to analyze the theory of probability through mathematical principles we must first discuss the formation of a mathematical model. While many types of data may be applied to a normal distribution curve, it should not be assumed that all sets of data conform to the curve. Data in the form of height and weight typically conform to the normal distribution curve. It is unlikely that data of a social nature would conform. Coin tossing is a form of data that does conform to the normal probability curve and we will use this type data for our discussion of distributions.

BINOMIAL

When we toss a coin three times, we list the possible outcomes as

TTT, TTH, THT, HTT, THH, HTH, HHT, HHH

Since each toss of the coin is independent and

$$P\{H\} = \frac{1}{2}$$

and

$$P\{T\} = \frac{1}{2}$$

then the probability of TTT is

$$P\{TTT\} = P\{T\}\,P\{T\}\,P\{T\}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$= \frac{1}{8}$$

The probability of each of the other outcomes is the same.

If we are interested in only the number of heads obtained we let x denote this and we may have 0, 1, 2, or 3 heads. In table form this is shown as

Outcome	TTT	TTH	THT	HTT	THH	HTH	HHT	HHH
x	0	1	1	1	2	2	2	3

Now, the probability of different values of x are

$P\{0\} = \frac{1}{8}$ (TTT occurs once in the eight chances)

$$P\{1\} = \frac{3}{8}$$

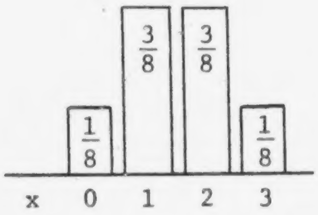
$$P\{2\} = \frac{3}{8}$$

$$P\{3\} = \frac{1}{8}$$

and in table form they appear as

x	0	1	2	3
P{x}	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

If we now put this information into a histogram the distributions will appear as



and this is a representation of a frequency distribution of the probabilities.

If we toss a die three times and let x become the number of twos showing, we may form the table, where T is the two and N is not a two, by writing

Outcome	NNN	TNN	NTN	NNT	TTN	TNT	NTT	TTT
x	0	1	1	1	2	2	2	3

Since the probabilities are different, that is

$$P\{T\} = \frac{1}{6}$$

and

$$P\{N\} = \frac{5}{6}$$

by use of the multiplication rule we may make a table as

Outcome	Probability
NNN	$\left(\frac{5}{6}\right)^3$
TNN	$\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$
NTN	$\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$
NNT	$\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$
TTN	$\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$
TNT	$\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$
NTT	$\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$
TTT	$\left(\frac{1}{6}\right)^3$

Now, since each of the groups of events are mutually exclusive—for example, where we have two tails and one non-tail (TTN, TNT, and NTT)—we add these together to find P{x}. Instead of adding the same thing three times we multiply by three. This gives us the table entry for P{x} where x = 2. We complete the table as follows:

x	0	1	2	3
P{x}	$\left(\frac{5}{6}\right)^3$	$3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$	$3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$	$\left(\frac{1}{6}\right)^3$

Then

$$P\{0\} = \left(\frac{5}{6}\right)^3 = \frac{125}{216}$$

$$P\{1\} = 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 = \frac{75}{216}$$

$$P\{2\} = 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) = \frac{15}{216}$$

$$P\{3\} = \left(\frac{1}{6}\right)^3 = \frac{1}{216}$$

If we let p be the probability of a two and q be the probability of not a two we may write

x	0	1	2	3
$P\{x\}$	q^3	$3q^2p$	$3qp^2$	p^3

This approach to frequency distributions is appropriate for small numbers of trials but when large numbers of trials are involved we rely upon the binomial distribution. This is given as

$$P\{x\} = \frac{n!}{x!(n-x)!}\, p^x q^{n-x}$$

Notice that

$$\frac{n!}{x!(n-x)!}$$

is really

$${}_nC_x$$

therefore

$$P\{x\} = {}_nC_x\, p^x q^{n-x}$$

In the die example the probability of one two showing is

$$P\{x\} = {}_nC_x\, p^x q^{n-x}$$

where

$$n = 3$$

$$p = \frac{1}{6}$$

$$x = 1$$

and

$$q = \frac{5}{6}$$

then

$$P\{1\} = {}_3C_1\, p^1 q^2$$

$$= \frac{3!}{1!(2)!}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$$

$$= 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$$

which agrees with our previous answer.

EXAMPLE: What is the probability of a one showing in three tosses of a die?

SOLUTION: Write

$$P\{x\} = {}_nC_x\, p^x q^{n-x}$$

and

$$x = 1$$

$$n = 3$$

$$p = \frac{1}{6}$$

$$q = \frac{5}{6}$$

therefore,

$$P\{1\} = {}_3C_1\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$$

$$= 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$$

$$= \frac{25}{72}$$

EXAMPLE: If a die is tossed five times, find the following probabilities $P\{x\}$: $x = 0, 1, 2, 3, 4, 5$.

SOLUTION: Write

$$P\{x\} = {}_nC_x\, p^x q^{n-x}$$

where

$x = 0, 1, 2, 3, 4, 5$

$n = 5$

$p = \frac{1}{6}$

$q = \frac{5}{6}$

then

$$P\{0\} = {}_5C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^5$$

$$= 1(1) \left(\frac{5}{6}\right)^5$$

$$= \frac{3125}{7776}$$

$$P\{1\} = {}_5C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^4$$

$$= 5 \left(\frac{1}{6}\right) \left(\frac{625}{1296}\right)$$

$$= \frac{3125}{7776}$$

$$P\{2\} = {}_5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3$$

$$= 10 \left(\frac{1}{36}\right) \left(\frac{125}{216}\right)$$

$$= \frac{1250}{7776}$$

$$P\{3\} = {}_5C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2$$

$$= 10 \left(\frac{1}{216}\right) \left(\frac{25}{36}\right)$$

$$= \frac{250}{7776}$$

$$P\{4\} = {}_5C_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^1$$

$$= 5 \left(\frac{1}{1296}\right) \left(\frac{5}{6}\right)$$

$$= \frac{25}{7776}$$

$$P\{5\} = {}_5C_5 \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^0$$

$$= 1 \left(\frac{1}{7776}\right) (1)$$

$$= \frac{1}{7776}$$

EXAMPLE: Find, in the preceding problem, the probability of at least four threes showing, that is

$$P\{x \geq 4\}$$

SOLUTION: We desire the probability of both $P\{4\}$ and $P\{5\}$. These are mutually exclusive, therefore we use the addition rule and write

$$P\{x \geq 4\} = P\{4\} + P\{5\}$$

$$= \frac{25}{7776} + \frac{1}{7776}$$

$$= \frac{26}{7776}$$

Notice that if we add all the probabilities together, we find the probability that any of the events will happen, which is,

$$P\{0\} + P\{1\} + P\{2\} + P\{3\} + P\{4\} + P\{5\}$$

$$= \frac{3125}{7776} + \frac{3125}{7776} + \frac{1250}{7776} + \frac{250}{7776} + \frac{25}{7776} + \frac{1}{7776}$$

$$= \frac{7776}{7776} = 1$$

This corresponds with our previous assumption that the sum of successful and failing events equals 1.

EXAMPLE: If a die is tossed eight times, what is the probability that a four will show exactly twice?

SOLUTION: Write

$$P\{x\} = {}_nC_x\, p^x\, q^{n-x}$$

where

$$n = 8$$

$$x = 2$$

$$p = \frac{1}{6}$$

$$q = \frac{5}{6}$$

then

$$P\{2\} = {}_8C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^6$$

$$= 28 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^6$$

EXAMPLE: If a die is tossed four times, what is the probability that a three will show at "most" two times?

SOLUTION: We must find the sum of $P\{0\}$, $P\{1\}$, and $P\{2\}$ therefore we write

$$P\{x\} = {}_nC_x\, p^x\, q^{n-x}$$

where

$$n = 4$$

$$x = 0, 1, 2$$

$$p = \frac{1}{6}$$

$$q = \frac{5}{6}$$

and

$$P\{0\} = {}_4C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^4$$

$$= 1\ (1) \left(\frac{625}{1296}\right)$$

$$= \frac{625}{1296}$$

$$P\{1\} = {}_4C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^3$$

$$= 4 \left(\frac{1}{6}\right) \left(\frac{125}{216}\right)$$

$$= \frac{500}{1296}$$

$$P\{2\} = {}_4C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2$$

$$= 6 \left(\frac{1}{36}\right) \left(\frac{25}{36}\right)$$

$$= \frac{150}{1296}$$

then

$$P\{x \le 2\} = P\{0\} + P\{1\} + P\{2\}$$

$$= \frac{625}{1296} + \frac{500}{1296} + \frac{150}{1296}$$

$$= \frac{1275}{1296}$$

In problems of a binomial nature, four properties are required. They are as follows:

1. The number of trials must be fixed.

2. Each trial must result in either a success or a failure.

3. The probability of successes must be identified.

4. All of the trials must be independent.

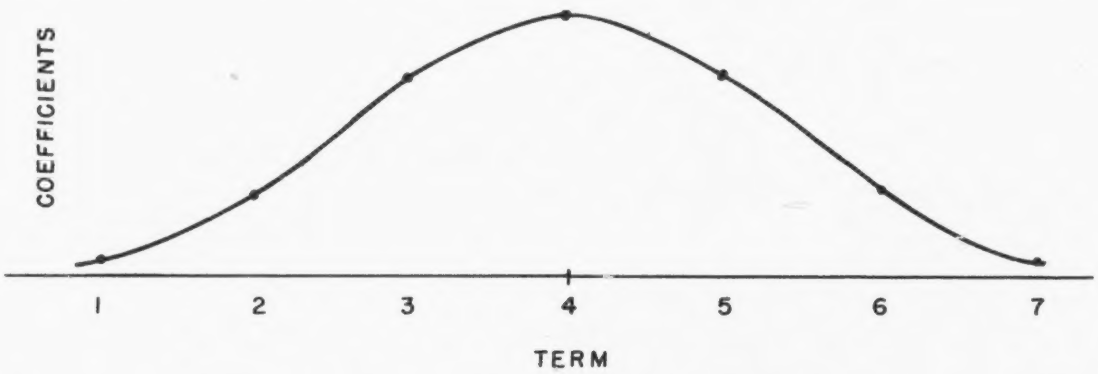


Figure 4-1.—Graph of coefficients of $(x + y)^n$, n = 6

In the previous problems we have made use of the formula

$$P\{x\} = {}_nC_x\, p^x\, q^{n - x}$$

When we plot the coefficient of p and q, that is,

$${}_nC_x$$

we find that the curve will resemble that shown in figure 4-1, depending on the value set for n.

NORMAL

When information from a large population is examined it will be found that there will be many deviations from the mean. Both positive and negative deviations will occur with nearly the same frequency. Also small deviations will occur more frequently than large deviations.

Many years ago an equation was determined by De Moivre and later it was applied more broadly to areas of measurement by Laplace and Gauss. This equation which exhibits the previous mentioned characteristics of a population, is

$$y = ke^{-h^2x^2}$$

We will treat h and k as constants of one. The equation then becomes

$$y = e^{-x^2}$$

where e is the base of the system of natural logarithms and equals approximately $\frac{11}{4}$. The equation, then, may be written as

$$y = \left(\frac{4}{11}\right)^{x^2}$$

When we assign values to x and derive the values for y a curve is developed as shown in figure 4-2.

It has been found, when the entire area under the curve equals one, that

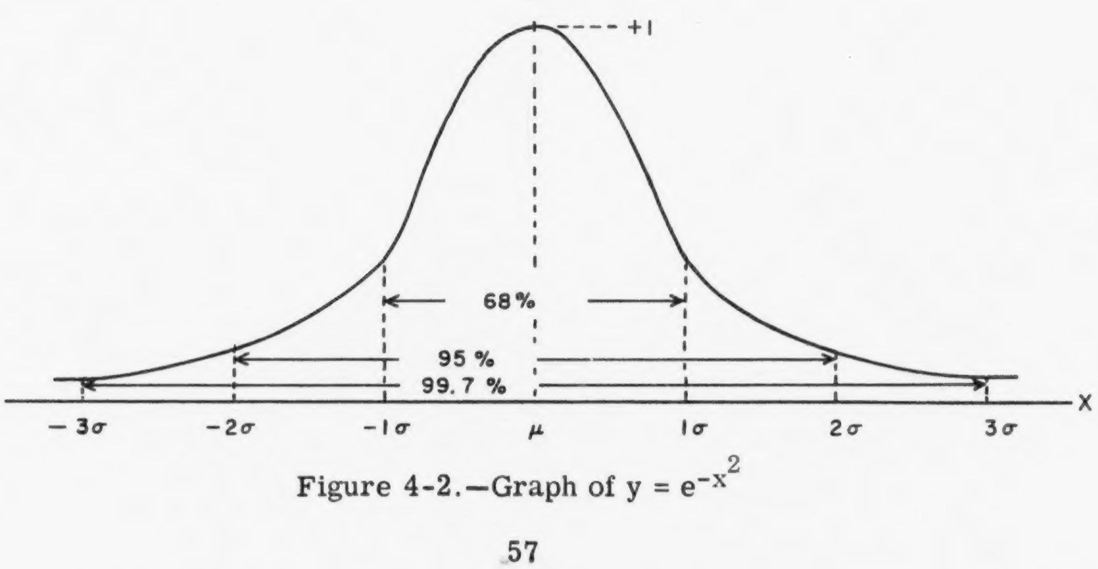


Figure 4-2.—Graph of $y = e^{-x^2}$

$$\mu \pm 1\sigma = 68 \text{ percent of the area}$$

$$\mu \pm 2\sigma = 95 \text{ percent of the area}$$

and

$$\mu \pm 3\sigma = 99.7 \text{ percent of the area}$$

(These areas are shown in figure 4-2.)

These areas also represent probabilities that a single variable will fall within these intervals. To use the table of areas under the normal curve the transformation formula

$$z = \frac{x - \mu}{\sigma}$$

must be applied.

We call the value z the standard normal deviate. This indicates the number of standard deviations the variable x is above or below the mean.

EXAMPLE: Find the area under the normal curve from z equal zero to z equal 1.5.

(This area is shown in figure 4-3.)

SOLUTION: In table 4-1 read down the first column to 1.5 then across to 0.00 and find 0.4332.

EXAMPLE: Find the area under the normal curve from z equal -0.4 to z equal 0.5.

(This area is shown in figure 4-4.)

SOLUTION: The table gives only areas from z equal zero to some positive value, therefore we rely on symmetry to find the area from z equal zero to z equal -0.4. Find the area from

$$z = 0$$

to

$$z = 0.4$$

as

$$0.1554$$

Then, the area from

$$z = 0$$

to

$$z = 0.5$$

as

$$0.1915$$

We now add the area

$$\begin{array}{r} 0.1554 \\ +\ 0.1915 \\ \hline 0.3469 \end{array} \text{ total area desired}$$

EXAMPLE: If we are given the distribution as shown in figure 4-5, what is the area between x equal 110 and x equal 125, if the mean is 120 and the deviation is 5? (We assume normal distribution.)

SOLUTION: We use

$$z = \frac{x - \mu}{\sigma}$$

to find the values of z which correspond to

$$x = 110$$

and

$$x = 125$$

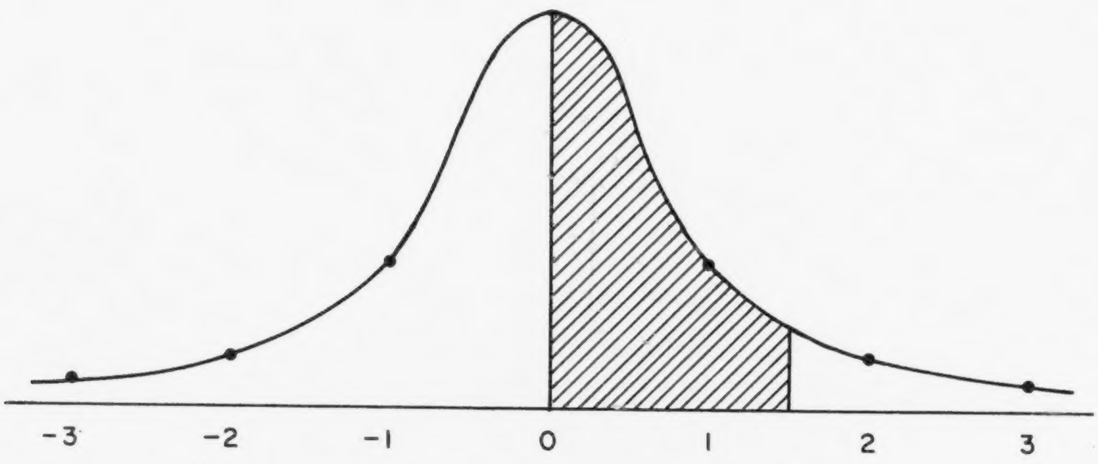


Figure 4-3.—Normal curve (z = 1.5).

Table 4-1.--Areas under the normal curve.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4986	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.5000	.5000	.5000
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

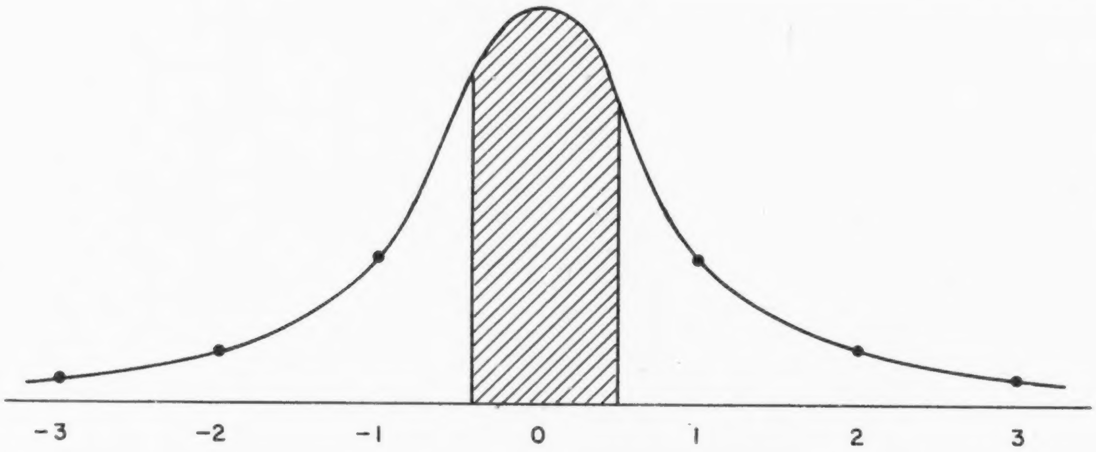


Figure 4-4.—Normal curve (z = 0.5).

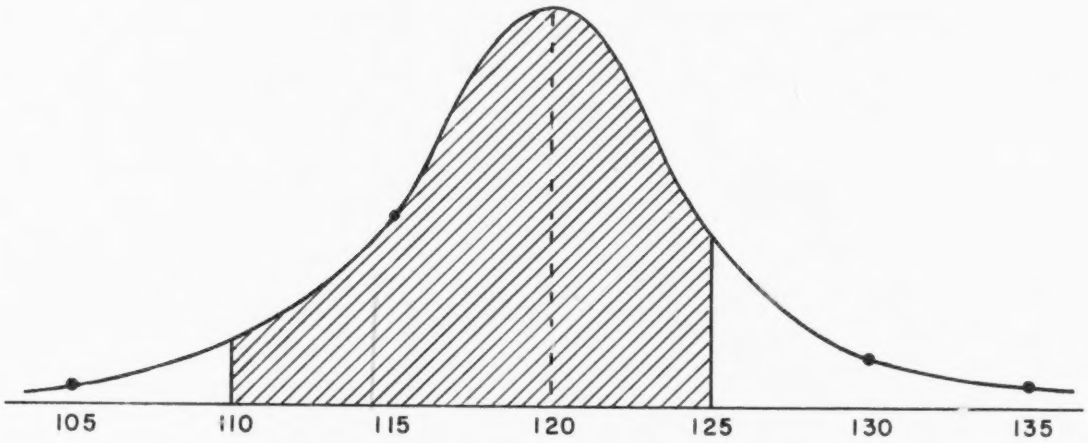


Figure 4-5.—Normal curve (x = 110 to 125).

We write

$$z = \frac{110 - 120}{5}$$

$$= -\frac{10}{5}$$

$$= -2$$

which corresponds to x equal 110. Then,

$$z = \frac{125 - 120}{5}$$

$$= \frac{5}{5}$$

$$= 1$$

which corresponds to x equal 125. By use of table 4-1 find the area from z equal zero to -2 to be 0.4772 and the area from z equal zero to +1 to be 0.3413. We then add the areas and find

$$\begin{array}{r} 0.4772 \\ +\ 0.3413 \\ \hline 0.8185 \end{array} \text{ total area desired}$$

While we have discussed areas under the curve, these areas are also the probabilities of occurrence; that is, in the preceding example, the probability that a single selected value will fall between 110 and 125 is 0.8185. These probabilities hold regardless of the value of the mean or standard deviation as long as the values form a normal distribution.

In problems of the previous type, it is advisable to draw a rough sketch of the curve to picture the areas or probabilities desired.

EXAMPLE: With a set of grades which form a normal distribution and have a mean of 70 and a standard deviation of 6, what is the probability that a grade selected at random will be higher than 78?

SOLUTION: Sketch the curve as shown in figure 4-6. The area or probability we desire is shaded. We find this probability by finding the probability of the grades above 70 then subtracting the probability of the grades from 70 to 78. We write

$$z = \frac{78 - 70}{6}$$

$$= \frac{8}{6}$$

$$= 1.33$$

In table 4-1

$$z = 1.33$$

is

$$0.4082$$

and the probability of grades above 70 is

$$0.5000$$

then

$$\begin{array}{rl} 0.5000 & \text{probability of grades above 70} \\ -\ 0.4082 & \text{probability of grades from 70 to 78} \\ \hline 0.0918 & \text{probability of grades above 78} \end{array}$$

Therefore, the probability that the grades selected will be higher than 78 is 0.0918.

POISSON

When we are faced with problems which have more outcomes than those of a binomial nature, that is 0 or 1, yes or no, or true or false, the Poisson distribution may be used. This distribution is defined with respect to a unit of measure where there may be several outcomes within the given unit of measure. It is used when the number of trials is extremely large and the probability of success in any trial is quite small.

The Poisson distribution is useful in quality control. An example may better explain this idea.

If we were to inspect boxes of resistors on a production line we might find 0, 1, 2, or more defective resistors per box. We could count the defective resistors but it would be impractical to count the number of good resistors. In this case we could use the Poisson distribution to find the probability of 0, 1, 2, or more defective resistors in a given box.

The formula for the Poisson probability is

$$P\{x\} = \frac{e^{-m}\, m^x}{x!} \quad \text{where } x = 0, 1, 2, 3, \ldots$$

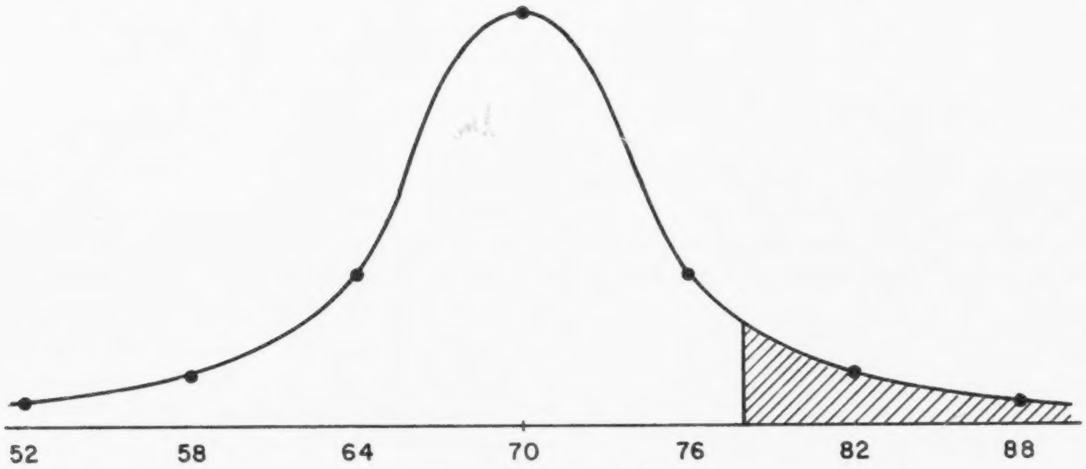


Figure 4-6.—Normal curve ($x \geq 78$).

x is the number of occurrences per unit, m is the average or mean of the occurrences per unit, and e is the base of the natural logarithms.

One requirement of the Poisson formula of probability is that the number of possible occurrences in any unit is large while the probability of a particular occurrence is small. A second requirement is that the particular occurrences in one unit do not influence the particular occurrences in another unit. Finally, the third requirement is that the average or mean must remain constant.

EXAMPLE: In our example problem suppose that on the average there were 2 defective resistors per box. What is the probability that there will be no defective resistors in a given box?

SOLUTION: Write

$$P\{x\} = \frac{e^{-m}\ m^x}{x!}$$

then

$$x = 0$$

$$e = 2.7$$

$$m = 2$$

therefore

$$P\{0\} = \frac{2.7^{-2}\ 2^0}{0!}$$

$$= \frac{2^0}{2.7^2}$$

$$= \frac{1}{7.29}$$

$$= 0.135$$

Notice that only the average or mean is a parameter of the Poisson distribution.

To continue our problem further, the probability of a random selected box having 3 defective resistors is

$$P\{3\} = \frac{e^{-2}\ 2^3}{3!}$$

$$= \frac{2.7^{-2}\ 8}{6}$$

$$= \frac{8}{7.29\,(6)}$$

$$= \frac{8}{43.74}$$

$$= 0.18$$

In the example given we have used the formula to determine the probabilities. Tables of the Poisson probabilities have been determined to alleviate this laborious process.

EXAMPLE: If on the average one person entered a store every 5 seconds and persons entered at random, what is the probability that in a selected 5-second period of time 3 people enter the store?

SOLUTION: Write

$$P\{x\} = \frac{e^{-m}\ m^x}{x!}$$

and

$$e = 2.7$$

$$x = 3$$

$$m = 1$$

therefore

$$P\{3\} = \frac{e^{-1}\ 1^3}{3!}$$

$$= \frac{2.7^{-1}\ (1)}{6}$$

$$= \frac{1}{2.7(6)}$$

$$= 0.06$$

Rather than solve this problem by use of the formula we could have used table 4-2. The table is formed with the value of x versus the value of m. In this problem we would have followed x equal 3 down until it fell in the row where m equals 1 which gives 0.061.

EXAMPLE: If aircraft arrive randomly at a field on the average of two every fifteen minutes, what is the probability that six aircraft will arrive in a given quarter-hour?

Table 4-2.--Poisson distribution (partial table).

$$f(x) = \frac{e^{-m}m^x}{x!}$$

x m	0	1	2	3	4	5	6	7	8	9
0.10	905	090	005							
0.15	861	129	010							
0.20	819	164	016	001						
0.25	779	195	024	002						
0.30	741	222	033	003						
0.40	670	268	054	007	001					
0.50	607	303	076	013	002					
0.60	549	329	099	020	003					
0.70	497	348	122	028	005	001				
0.80	449	359	144	038	008	001				
0.90	407	366	165	049	011	002				
1.00	368	368	184	061	015	003	001			
1.10	333	366	201	074	020	004	001			
1.20	301	361	217	087	026	006	001			
1.30	273	354	230	100	032	008	002			
1.40	247	345	242	113	039	011	003	001		
1.50	223	335	251	126	047	014	004	001		
1.60	202	323	258	138	055	018	005	001		
1.70	183	311	265	150	063	022	006	001		
1.80	165	298	268	161	072	026	008	002		
1.90	150	284	270	171	081	031	010	003	001	
2.00	135	271	271	180	090	036	012	003	001	
2.10	122	257	270	189	099	042	015	004	001	
2.20	111	244	268	197	108	048	017	005	002	
2.30	100	231	265	203	117	054	021	007	002	
2.40	091	218	261	209	125	060	024	008	002	001
2.50	082	205	257	214	134	067	028	010	003	001
2.60	074	193	251	218	141	074	032	012	004	001

SOLUTION: Write

$$P\{x\} = \frac{e^{-m}\,m^x}{x!}$$

where

$$x = 6$$

$$m = 2$$

then, by use of table 4-2 find that

$$P\{6\} = 0.012$$

EXAMPLE: A certain type aircraft averages 1.1 failures requiring ground maintenance for every 24 hour day. What is the probability of a squadron having 4 or more aircraft grounded for maintenance on a particular day?

SOLUTION: We must find the probabilities of x equal 4, 5, 6, ... By use of table 4-2 we find

$$P\{4\} = 0.020$$

$$P\{5\} = 0.004$$

$$P\{6\} = 0.001$$

$$P\{7\} = 0.000$$

therefore,

$$\begin{aligned} P\{x \geq 4\} &= 0.020 + 0.004 + 0.001 \\ &= 0.25 \end{aligned}$$

Poisson to Binomial Approximation

The Poisson approximation to the binomial holds if the value of n is large and the value of p is small. Generally, if the ratio of n to p is near 1000 we can use the Poisson to approximate the binomial; that is, if n is greater than 10 and p is less than 0.01 the ratio of n/p is 1000.

In order to approximate a binomial we set np equal to m and use the Poisson table. That is, if we are sampling 50 items which have a probability of defect of 0.05 on the average, we write

$$n = 50$$

$$p = 0.05$$

and

$$np = m$$

$$(50)(0.05) = 2.5$$

then we may estimate the probability of a number of defects by using table 4-2. In this case

$$P\{0\} = 0.082$$

$$P\{1\} = 0.205$$

$$P\{2\} = 0.257$$

Normal to Binomial Approximation

When n was large and p was small, the Poisson distribution could be used to approximate the binomial. When n is large and p is neither small nor large (not close to 1 or 0), we can use the normal to approximate the binomial. To use this approximation the product of np should be equal to or greater than 5; that is, if n were 20 then

$$\begin{aligned} np &\geq 5 \\ 20p &\geq 5 \\ p &\geq \frac{5}{20} \\ &\geq \frac{1}{4} \\ &\geq 0.25 \end{aligned}$$

With n equal to 20, p should be 0.25 or greater in order that the distribution be nearly normal. Steps in using the approximation are:

1. Let np equal μ.
2. Let $\sqrt{npq}$ equal σ.
3. If finding the probabilities of some number or less successes add 0.5 to x and if finding more successes subtract 0.5 from x (this is due to the binomial being a discrete distribution).
4. Use the normal table.

EXAMPLE: If the probability of a defective item is 0.2 and we use a sample of 500 items from a large population, what is the probability of 30 or more defective items?

SOLUTION: Write

$$\begin{aligned} \mu &= np \\ &= 500(0.2) \\ &= 100 \end{aligned}$$

and

$$\begin{aligned} \sigma &= \sqrt{npq} \\ &= \sqrt{500\ (0.2)\ (0.8)} \\ &= \sqrt{80} \\ &= 8.9 \end{aligned}$$

Then,

$$\begin{aligned} z &= \frac{x - \mu}{\sigma} \\ &= \frac{29.5 - 10}{8.9} \\ &= 2.18 \end{aligned}$$

Using table 4-1 find that the probability

$$\begin{aligned} P\{z > 2.18\} &= 0.5000 - 0.4854 \\ &= 0.0146 \end{aligned}$$

In this same example, what is the probability of exactly 30 defective items? Since the probability of more than 30 defective items in the binomial distribution is the same as 30.5 defective items in the normal distribution we use x equal 30.5. (Again, this is due to the binomial being a discrete distribution.) Then

$$\mu = 10$$

$$x = 30.5$$

$$\sigma = 8.9$$

and we write

$$z = \frac{30.5 - 10}{8.9}$$

$$= \frac{20.5}{8.9}$$

$$= 2.3$$

Using table 4-1 find that

$$P\{z > 2.3\} = 0.5000 - 0.4893$$

$$= 0.0107$$

The probability of exactly 30 defective items is

$$P\{2.18 < z < 2.3\} = 0.0146 - 0.0107$$

$$= 0.0039$$

The preceding example is illustrated in figure 4-7.

EXAMPLE: If the probability of success in a single try is $\frac{1}{3}$, what is the probability of at least 6 successes in 12 tries?

SOLUTION: Write

$$n = 12$$

$$p = \frac{1}{3}$$

$$q = \frac{2}{3}$$

and

$$\mu = np$$

$$= 12\left(\frac{1}{3}\right)$$

$$= 4$$

and

$$\sigma = \sqrt{npq}$$

$$= \sqrt{12 \cdot \frac{1}{3} \cdot \frac{2}{3}}$$

$$= \sqrt{\frac{24}{9}}$$

$$= 1.63$$

then

$$z = \frac{x - \mu}{\sigma}$$

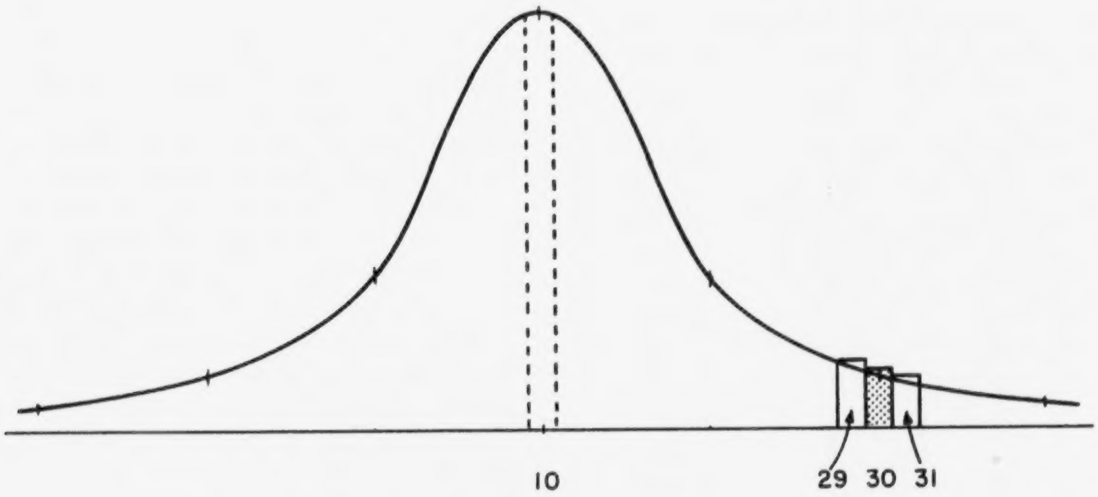


Figure 4-7.—Normal curve (x = 30.5).

(where 0.5 is subtracted from x)

$$z = \frac{6 - 0.5 - 4}{1.63}$$
$$= \frac{5.5 - 4}{1.63}$$
$$= \frac{1.5}{1.63}$$
$$= 0.92$$

and by use of table 4-1 find that

$$P\{z > 0.92\} = 0.5000 - 0.3212$$
$$= 0.1788$$

INTERPRETATION OF STANDARD DEVIATION

We have discussed distribution and how special cases may approximate the normal distribution. When a normal distribution is determined the standard deviation (σ) enables us to determine characteristics of the distribution. It has been found that 99.7 percent of all items of a normal distribution fall within three standard deviations of the mean.

In table 4-3 the f column is the fractional part of the standard deviation and the A column is the area under the normal curve for ± the fractional part indicated. That is, the area under the normal curve which falls within $\pm 1.3\ \sigma$ is 0.807 or 80.7 percent.

Table 4-3.—Values for ± standard deviations.

f	A	f	A
0.1	0.080	1.6	0.891
0.2	0.159	1.7	0.911
0.3	0.236	1.8	0.928
0.4	0.311	1.9	0.943
0.5	0.383	2.0	0.955
0.6	0.451	2.1	0.964
0.7	0.516	2.2	0.972
0.8	0.576	2.3	0.979
0.9	0.632	2.4	0.984
1.0	0.683	2.5	0.988
1.1	0.729	2.6	0.991
1.2	0.770	2.7	0.993
1.3	0.807	2.8	0.995
1.4	0.838	2.9	0.996
1.5	0.866	3.0	0.997

In a set of data which is normally distributed and has a mean of 78 and a standard deviation of 6; that is,

$$\bar{x} = 78$$

and

$$\sigma = 6$$

the area under the curve between $+0.5\ \sigma$ and $-0.5\ \sigma$ is 0.383. This means that 38.3 percent of the data will fall within the range of

$$\bar{x} + 0.5\ \sigma \text{ and } \bar{x} - 0.5\ \sigma$$

or within

$$78 + 0.5(6) \text{ and } 78 - 0.5(6)$$

which is

$$81 \text{ and } 75$$

An interpretation of the previous statements is that $0.5\ \sigma$ of the grades fall above the mean of 78 and $0.5\ \sigma$ of the grades fall below the mean.

STANDARD SCORES

In many cases it is necessary to combine scores or achievement ratings from different tests into a single grade or rating. To combine raw scores from different tests is not statistically sound unless the means and standard deviations of the different tests are the same. The probability of this occurring is quite small; therefore, we resort to the standard score.

When we change raw scores into standard scores, we assume the raw scores form a normal distribution. The standard scores are expressed in standard deviations with a mean of zero and a standard deviation of one.

Standard scores may be added and averaged with equal weight given to each different score. As an example, we may desire an overall merit rating of persons who were given several tests. While some would score high in a particular field they may score low in others and furthermore, the difficulty of one test may vary from the difficulty of another. What we desire is a rating for each area tested in a form that can be compared with ratings of other areas tested. For this we use the standard score.

The standard score, for a particular test, is determined by the formula

$$\text{Standard Score} = \frac{\text{Raw Score - Mean}}{\text{Standard Deviation}}$$

$$= \frac{x - \bar{x}}{\sigma}$$

EXAMPLE: If the raw scores on an examination were 68, 70, 73, 76, 81, 90, and 95, convert these to standard scores.

SOLUTION: We must find $\bar{x}$ and σ. Write

$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$$

$$= \frac{553}{7}$$

$$= 79$$

and

$$\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$= \sqrt{\frac{1}{7} (628)}$$

$$= \sqrt{89.7}$$

$$\simeq 9.4$$

Then write

x	x - x	$\frac{x - \bar{x}}{\sigma}$	Standard Score
95	16	16 ÷ 9.4	1.65
90	11	11 ÷ 9.4	1.16
81	2	2 ÷ 9.4	0.21
76	-3	-3 ÷ 9.4	-0.32
73	-6	-6 ÷ 9.4	-0.63
70	-9	-9 ÷ 9.4	-0.95
68	-11	-11 ÷ 9.4	-1.16

The positive standard scores indicate that factor of standard deviations above the mean of the raw scores and negative scores indicate that factor of standard deviations below the mean of the raw scores.

If a person, on five different tests, had standard scores of +1.32, -0.93, +2.03, +0.20, and -1.20, his average standard score would be

$$\begin{array}{r} +1.32 \\ -0.93 \\ +2.03 \\ +0.20 \\ -1.20 \\ \hline +1.43 \div 5 \end{array}$$

or

$$\frac{+1.43}{5} = +0.28$$

which indicates an achievement of 0.28 standard deviations above the mean.

When standard scores with a standard deviation of one and a mean of zero are determined they involve the use of positive and negative decimals. To eliminate the use of negative scores and decimals a linear transformation may be made by the use of a greater mean and a greater standard deviation.

An example of this type transformation is made on the Graduate Record Examination. The scores on the G.R.E. are expressed using a standard deviation of 100 and a mean of 500.

To change a standard score to a corrected standard score with a mean of 500 and a standard deviation of 100 write

$$\text{Standard Score} = \frac{x - \bar{x}}{\sigma} (\sigma_c) + \bar{x}_c$$

where

σ_c = new standard deviation (100 in our case)

and

$\bar{x}_c$ = new mean (500 in our case)

Therefore, if the standard score is 0.6 then the corrected standard score is

$$0.6\ (\sigma_c) + \bar{x}_c$$

$$= 0.6\ (100) + 500$$

$$= 60 + 500$$

$$= 560$$

EXAMPLE: Change the standard score of -1.3 to a distribution with a mean of 500 and a standard deviation of 100.

SOLUTION: Write

$$\text{Standard Score (S.S.)} = -1.3$$

$$\sigma_c = 100$$

$$\bar{x}_c = 500$$

then the corrected standard score is

$$-1.3\ (100) + 500$$

$$= -130 + 500$$

$$= 370$$

PROBLEMS: Find the corrected standard score in the distribution indicated of the given standard scores.

1. 1.3 in distribution with $\bar{x}_c = 50$ and $\sigma_c = 10$.

2. -2.4 in distribution with $\bar{x}_c = 50$ and $\sigma_c = 10$.

3. -0.3 in distribution with $\bar{x}_c = 100$ and $\sigma_c = 20$.

ANSWERS:

1. 63

2. 26

3. 94

SAMPLING

When sampling a population one should be careful not to allow any preventable bias from becoming part of the sample data.

If a sample of the population were being taken to determine the average height of 16-year-old boys a bias would be introduced if the sample came from basketball players because they would probably be the tallest boys in the population. Bias in a sample will cause the predicted results to be in error.

A classic example of bias in a sample was encountered in the Literary Digest poll conducted in 1936 which resulted in the prediction that Landon would be elected President. Roosevelt was elected and upon investigation of the methods of sampling it was found that the sample was taken from persons who had a telephone and from those who had an automobile. This resulted in only a certain income group being polled which introduced a bias.

In order to select a random sample which will have little bias, the conscious selection of sample must not enter into the selection process. The use of a table of random numbers will aid in the removal of bias from the sample.

One use of a sample is to make a prediction about the population. It is known that the means of samples follow the normal distribution even though the population may vary somewhat from a normal distribution. Our predictions, from a sample, about the population are made with a certain level of confidence. The standard error of the mean allows us to give a level of confidence concerning the sample mean.

The standard error of the mean is

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

where N is the number in the sample and σ is the standard deviation. Since the sample means follow the normal distribution curve we may use table 4-1. Note that N should be greater than 25.

We have previously determined that $\bar{x} \pm 1\sigma$ contains about 68 percent of the items. We determine this from table 4-1 by finding 1.0 and reading .3413. Our table gives only the positive values above the mean; therefore, we double this figure (because of symmetry) and find

$$\begin{array}{l} .3413 \\ \underline{.3413} \\ .6826 \text{ or } 68 \text{ percent} \end{array}$$

When we speak of some level of confidence, such as 5 percent, we mean there is less than a 5 percent chance that the sample mean will differ from the population mean by some given amount.

We will use an example to illustrate the preceding statements. If we desired to find the average height of 16-year-old boys in a certain city we could select at random 36 boys and measure their heights. From the data collected suppose we find that $\bar{x} = 66$ inches and $\sigma = 2$ inches. Our problem now is to estimate the mean and standard deviation of the population. The standard error of the mean is

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

$$= \frac{2}{\sqrt{36}}$$

$$= \pm \frac{2}{6}$$

$$= \pm\ 0.3$$

We may now state that there is a .68 probability that the population mean is within the range of

$$\bar{x} \pm 1\sigma_{\bar{x}}$$

or

$$66 \pm (1)(.3)$$

or

$$66.3 \text{ and } 65.7$$

Our level of confidence is 32 percent which means that there is less than 32 percent chance that the sample mean differs from the true mean by $\bar{x} \pm 1\sigma_x$.

If we increase the size of the sample we will obtain a smaller range for our confidence level of 32 percent. Suppose we increase our sample from 36 to 100. Then

$$\bar{x} = 66$$

$$\sigma = 2$$

and

$$\sigma_{\bar{x}} = \frac{2}{\sqrt{100}}$$

$$= \pm\ \frac{2}{10}$$

$$= 0.2$$

and we now have a .68 probability that the population mean is within the range of

$$\bar{x} \pm 1\sigma_{\bar{x}}$$

or

$$66 \pm (1)(.2)$$

or

$$66.2 \text{ and } 65.8$$

as compared to

$$66.3 \text{ and } 65.7$$

when our sample number was 36.

In the preceding discussion it must be understood that the population should be large compared to the sample size.

EXAMPLE: Given a sample where $\bar{x} = 70$ and $\sigma = 4$ and $N = 100$ find the range about the mean which gives a 5 percent confidence level.

SOLUTION: We want a .95 probability that the true mean will be within a range to be determined.

Write

$$\bar{x} = 70$$

$$\sigma = 4$$

$$N = 100$$

and

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

$$= \frac{4}{\sqrt{100}}$$

$$= \pm\ \frac{4}{10}$$

$$= \pm\ .4$$

We desire .95 probability; therefore, we divide by 2 because our table values are for one-half the total area.

$$\frac{.95}{2} = .475$$

and find in table 4-1 that .475 is given for a factor of $\sigma_{\overline{x}}$ of 1.96.
Then, the range we desire is

$$\overline{x} \pm 1.96\,\sigma_{\overline{x}}$$

or

$$70 \pm 1.96\ (.4)$$

which is

$$70 \pm .784$$

or

$$70.784 \text{ and } 69.216$$

Thus, the probability is .95 that the true mean lies in the range

$$70.784 \text{ and } 69.216$$

PROBLEM: A random sample of forty resistors from an extremely large supply reveals a mean resistance of 1000 ohms and a standard deviation of 5 ohms. Find the standard error of the mean. What is the range about the mean of the sample which will give a .90 probability that the true resistance value of all the resistors will fall within it?

ANSWER:

(a) $\sigma_{\overline{x}} = .8$

(b) 1000 ± 1.32 or 1001.32 and 998.68

CHAPTER 5
NUMBER SYSTEMS

A number system is a set of symbols or characters which stand for numbers and are used for counting, adding, subtracting, etc. All number systems are related to each other by means of symbols, referred to as digits, although some number systems do not contain all of the same digits of another system. The decimal system will be used as a basis for our discussion of the other number systems.

Two important discoveries have been made since ancient times which greatly simplify the operations of numbers. These are the numeral zero and the principle of place value. The principle of place value consists of giving a numeral a value that depends on its position in the entire number. For example, in the numbers 463, 643, and 364, the 4 has a different value in each by virtue of its position or place value. In the first number it means 4 hundreds, in the second number it means 4 tens, and in the last number it means 4 ones. The zero is used in cases where a number does not have a value for a particular place value; that is, in the number 306 there is no value for tens and the 0 indicates this. We say it is used as a place holder. It would be difficult to express the number three hundred six without using the zero.

We will discuss systems of numbers, basic operations in these systems, and the processes used to convert from one system to another system.

SYSTEMS

Systems of numbers are identified by their radix or base. The base of a number system is a number indicating how many characters or symbols it possesses, including zero. In the Hindu-Arabic system, the system we use and call the decimal system, there are ten symbols or digits. These are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The proper names along with their corresponding bases for several of the different systems are listed as follows:

Base	Name
2	Binary
3	Ternary
4	Quaternary
5	Quinary
6	Senary
7	Septenary
8	Octanary (Octal)
9	Novenary
10	Decimal
12	Duodecimal
16	Sexadecimal (Hexadecimal)
20	Vicenary (Vigesimal)
60	Sexagesimal

DECIMAL SYSTEM

When we count in base ten, or in the decimal system, we begin with 0 and count through 9. When we reach 9 and attempt to count one more unit, we rely on place value; that is, we say 9 and one more is ten. We show this by writing

$$10$$

which in place value notation means one group of ten and no groups of one. While we have a proper name for 10, we could call this "one-zero" in base ten.

The place value chart for base ten is formed by writing the base in columns and then assigning exponents to the base of each column in ascending order from the right to the left starting with zero; that is,

$$\cdots \quad (10)^4 \quad (10)^3 \quad (10)^2 \quad (10)^1 \quad (10)^0$$

The column with value $(10)^0$ is the column to the left of the decimal point. When we write a number in base ten, we do not use a subscript, whereas in other bases we do make the identification of the base by the subscript. In the number 3762, in the place value columns we have

$(10)^3$	$(10)^2$	$(10)^1$	$(10)^0$
3	7	6	2

which means 3 groups of $(10)^3$ or 3000, 7 groups of $(10)^2$ or 700, 6 groups of $(10)^1$ or 60, and 2 groups of $(10)^0$ or 2. Recall that any number raised to the zero power is equal to one. The previous number may be written in polynomial form as

$$3(10)^3 + 7(10)^2 + 6(10)^1 + 2(10)^0$$

In general, then, if B is the base of a system the place value columns are as follows:

$$\cdots B^5 \quad B^4 \quad B^3 \quad B^2 \quad B^1 \quad B^0 \cdot B^{-1} \quad B^{-2} \cdots$$

although we will discuss only the values of a base for B^0 and larger; that is, the whole numbers and not fractions.

QUINARY SYSTEM

When counting in the base five (Quinary) system, we must limit ourselves to the use of only the digits in the system. These digits are 0, 1, 2, 3, and 4. If we start counting with zero, we have

$$0_5, 1_5, 2_5, 3_5, 4_5, 10_5, 11_5, 12_5, 13_5, 14_5, 20_5, \ldots$$

We identify our system by the subscript of 5. Therefore,

$$10_5$$

is really, in place value columns,

...	5^2	5^1	5^0
		1	0

which means one group of 5 and no groups of 1. Counting is shown as

...	5^2	5^1	5^0
			0
			1
			2
			3
			4
		1	0
		1	1
		1	2
		1	3
		1	4
		2	0
		2	1
		2	2
		2	3
		2	4
		3	0

The number 342_5 means 3 groups of 5^2 plus 4 groups of 5^1 plus 2 groups of 5^0. In place value notation it is

5^2	5^1	5^0
3	4	2

BINARY SYSTEM

The only digits in the binary system are 0 and 1. We form the place value columns by starting at the radix, or binary, point (analogous to "decimal point") and assign the base (2) the exponent of 0. Then, moving to the left, we increase the exponent by one for each place value column; that is, each column to the left is a multiple of the one on the right. We write

$$\cdots 2^7 \quad 2^6 \quad 2^5 \quad 2^4 \quad 2^3 \quad 2^2 \quad 2^1 \quad 2^0$$

When we count in base 2, we start with zero and proceed as follows:

...	2^2	2^1	2^0
			0
			1
		1	0
		1	1
	1	0	0
	1	0	1
	1	1	0

Notice that the last number counted, that is, 110_2, is one group of 2^2, one group of 2^1, and no groups of one. Written in polynomial form it is

$$1(2)^2 + 1(2)^1 + 0(2)^0$$

In a binary number, each digit has a particular power of the base associated with it. This power of the base is called the positional notation. This is sometimes called the weighting value and depends upon the position of its digit. In the number 101_2 the weighting value of the leftmost digit is $(2)^2$. Notice that the weighting value of any digit is the base raised to a power which is equal to the number of digits to the right of the digit being discussed. That is, in the number 101101_2, the leftmost digit has five digits to its right; therefore, the weighting value of the leftmost digit is $(2)^5$. This weighting value is also called the position coefficient.

OCTAL SYSTEM

In the octal system the digits are 0, 1, 2, 3, 4, 5, 6, and 7. The place value columns, indicated by the weighting values, are

$$\cdots 8^5 \quad 8^4 \quad 8^3 \quad 8^2 \quad 8^1 \quad 8^0$$

The number 2307_8 means 2 groups of 8^3, 3 groups of 8^2, 0 groups of 8^1, and 7 groups of 8^0.

Counting in base eight is as follows:

0
1
2
3
4
5
6
7
1 0
1 1
1 2
1 3
⋮

Notice that after 7_8 we write 10_8 which is read "one-zero" and not "ten."

The number 2307_8 in polynomial form is

$$2(8)^3 + 3(8)^2 + 0(8)^1 + 7(8)^0$$

In general we may express a number as

$$a_1(r)^{n-1} + a_2(r)^{n-2} + \cdots + a_n(r)^{n-n}$$

where a is any digit in the number system, r is the radix or base, and n is the number of digits to the left of the radix or base point. (In the decimal system the radix point is the decimal point.) Again, the number 2307_8 is

$$a_1(r)^{n-1} + a_2(r)^{n-2} + a_3(r)^{n-3} + a_4(r)^{n-4}$$

or

$$2(8)^3 + 3(8)^2 + 0(8)^1 + 7(8)^0$$

where

$n = 4$
$r = 8$
$a_1 = 2$
$a_2 = 3$
$a_3 = 0$
$a_4 = 7$

DUODECIMAL SYSTEM

In the duodecimal (base 12) system we must create two new symbols. These symbols are needed because as we count in base 12; that is,

0
1
2
3
4
5
6
7
8
9

we find we cannot write the next number as 10 because this would indicate one group of twelve and no groups of one and we have not counted that high. We therefore write

⋮
7
8
9
t
e

where t indicates ten and e indicates eleven in the counting process familiar to us. To count further we write

⋮
8
9
t
e
10
11
12
⋮
1t
1e
20
⋮

where 12 means one group of $(12)^1$ and two groups of $(12)^0$. The number 1t means one group of $(12)^1$ and ten groups of $(12)^0$. The place value columns for base twelve are:

$$\cdots (12)^4 \quad (12)^3 \quad (12)^2 \quad (12)^1 \quad (12)^0$$

The number $2t9e6_{12}$ is

$$2(12)^4 + t(12)^3 + 9(12)^2 + e(12)^1 + 6(12)^0$$

HEXADECIMAL SYSTEM

This system has a base of sixteen; therefore, we need sixteen different symbols. The symbols we use are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F.

The number $9A6F_{16}$ means

$$9(16)^3 + A(16)^2 + 6(12)^1 + F(16)^0$$

In this system A is our familiar ten and F is fifteen. Note that the symbol 10_{16} is not "ten" but instead means "sixteen." The number systems discussed to this point may be compared by use of figure 5-1. In this table assume you are counting objects in each of the systems. Notice that when our count reaches object number eleven in base ten, this object is identified by 1011_2, 21_5, 13_8, e_{12}, and B_{16} in the various bases.

OPERATIONS

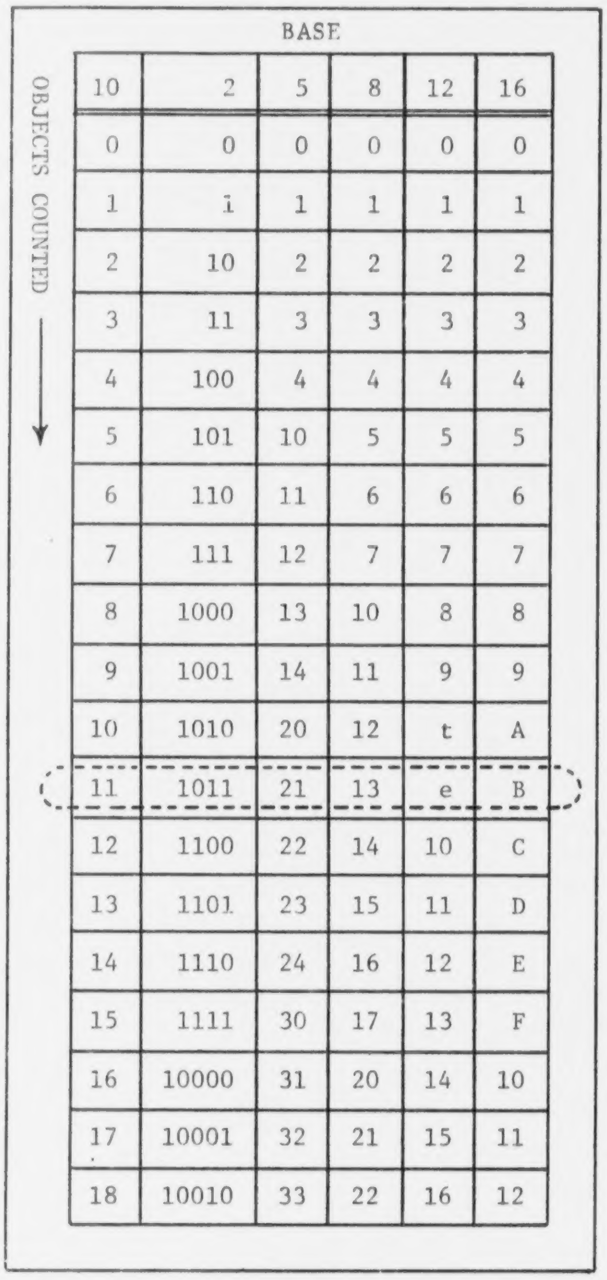
The operations we will discuss, for various bases, will be the basic arithmetic operations of addition, subtraction, multiplication, and division. Ease in performing these operations will facilitate ease of understanding conversions from one base to another base which will be discussed later in this chapter.

ADDITION

In general, the rules of arithmetic apply to any number system. Each system has a unique digit addition and digit multiplication table. These tables will be discussed with each system.

Decimal

Addition facts in base ten are shown in figure 5-2. The sign of operation is given in the upper left corner. The addends are indicated by row A and column B. The sums are shown



BASE
OBJECTS COUNTED

10	2	5	8	12	16
0	0	0	0	0	0
1	1	1	1	1	1
2	10	2	2	2	2
3	11	3	3	3	3
4	100	4	4	4	4
5	101	10	5	5	5
6	110	11	6	6	6
7	111	12	7	7	7
8	1000	13	10	8	8
9	1001	14	11	9	9
10	1010	20	12	t	A
11	1011	21	13	e	B
12	1100	22	14	10	C
13	1101	23	15	11	D
14	1110	24	16	12	E
15	1111	30	17	13	F
16	10000	31	20	14	10
17	10001	32	21	15	11
18	10010	33	22	16	12

Figure 5-1.—Number bases.

in the array C. To find the sum C of A + B locate the addends A and B. The sum C will be located where A and B intersect. The commutative principle causes the table to be symmetrical with respect to the diagonal with a negative slope. This is shown by the dotted line.

Quinary

Quinary addition facts are shown in figure 5-3. The table is symmetrical as was the decimal table. In adding 3_5 and 4_5 one should mentally add these and find the sum of 7. There is no symbol for 7 in base five, but one group of five and two groups of one will indicate this sum. That is,

$$\begin{array}{r} 4_5 \\ +\ \ 3_5 \\ \hline 1\ 2_5 \end{array}$$

where 12_5 is really

	5^1	5^0	
	1	2	5

In the decimal system a carry of ten is made, but in the quinary system a carry of five is made.

EXAMPLE: Add 324_5 and 433_5.
SOLUTION: Write

$$\begin{array}{r} 324_5 \\ +\ 433_5 \\ \hline \end{array}$$

Then, 4_5 plus 3_5 is 12_5 therefore, write

$$\begin{array}{r} 324_5 \\ +\ 433_5 \\ \hline 2_5 \end{array}$$

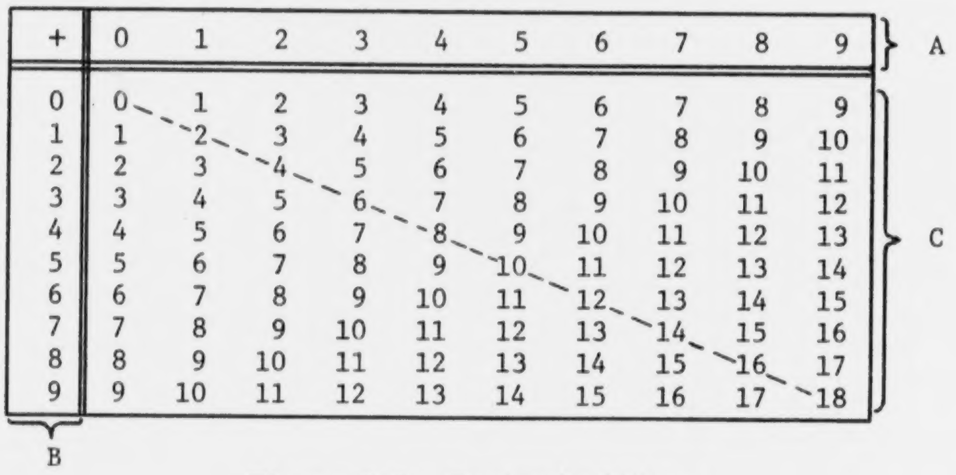
with a carry of one group of five.
Then, 2_5 plus 3_5 is 10_5 and the carry brings the total to 11_5 so we write

$$\begin{array}{r} 324_5 \\ +\ 433_5 \\ \hline 12_5 \end{array}$$

with a carry of one group of $(5)^2$. This gives 3_5 plus 4_5 equals 12_5 and the carry of one brings the total to 13_5. Therefore, we have

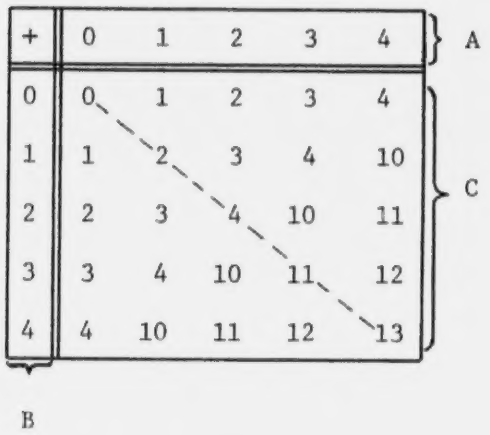
$$\begin{array}{r} 324_5 \\ +\ 433_5 \\ \hline 1312_5 \end{array}$$

EXAMPLE: Add 3042_5 and 4323_5.
SOLUTION: Write



+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Figure 5-2.—Decimal addition.



+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

Figure 5-3.—Quinary addition.

$$\begin{array}{r} 3042_5 \\ +\ 4323_5 \\ \hline \end{array}$$

Then, in steps, with the carry indicated, we have

$$\begin{array}{r} 1 \\ 3042_5 \\ +\ 4323_5 \\ \hline 0_5 \end{array}$$

$$\begin{array}{r} 1 \\ 3042_5 \\ +\ 4323_5 \\ \hline 20_5 \end{array}$$

$$\begin{array}{r} 0 \\ 3042_5 \\ +\ 4323_5 \\ \hline 420_5 \end{array}$$

$$\begin{array}{r} 3042_5 \\ +\ 4323_5 \\ \hline 12420_5 \end{array}$$

This process is quite satisfactory until we try to add, in base five, the following (omit subscripts for simplicity):

$$\begin{array}{r} 44 \\ 43 \\ 24 \\ 34 \\ 24 \\ 33 \\ 23 \\ +\quad 13 \\ \hline \end{array}$$

We add the 5^0 column and find we have a sum of 28 or 103_5. (103_5 may be found by counting 28 objects.) Our problem is, what do we carry? In this case we carry 10_5 which is 5. Then the next column sum is 26 which is 101_5. Therefore, we write the sum as

$$1013_5$$

PROBLEMS: Add the following base five numbers.

1. $$\begin{array}{r} 302 \\ +\ 443 \\ \hline \end{array}$$

2. $$\begin{array}{r} 40321 \\ +\ 43444 \\ \hline \end{array}$$

3. $$\begin{array}{r} 434 \\ 424 \\ 143 \\ 344 \\ 204 \\ 432 \\ 443 \\ +\ 342 \\ \hline \end{array}$$

ANSWERS:

1. 1300_5
2. 134320_5
3. 11041_5

Binary

Binary addition facts are shown in figure 5-4. Notice that a binary digit has only two possible values, 0 and 1. A carry of two is involved in binary addition. That is, when we add one and one the sum is two, but we have no two so we write 10_2 which indicates one group of $(2)^1$ and no group of one.

EXAMPLE: Add 1011_2 and 1101_2.
SOLUTION: Write

$$\begin{array}{r} 1011_2 \\ +\ 1101_2 \\ \hline \end{array}$$

Then, one and one are two, but "two" is

$$10_2$$

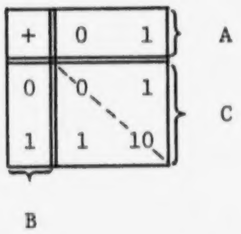


Figure 5-4.—Binary addition.

so we write

$$\begin{array}{r} 1 \\ 1011_2 \\ +\ 1101_2 \\ \hline 0 \end{array}$$

and carry a one. The following steps, with the carry indicated, show the completion of our addition.

$$\begin{array}{r} 1 \\ 1011_2 \\ +\ 1101_2 \\ \hline 0_2 \end{array}$$

$$\begin{array}{r} 1 \\ 1011_2 \\ +\ 1101_2 \\ \hline 00_2 \end{array}$$

$$\begin{array}{r} 1 \\ 1011_2 \\ +\ 1101_2 \\ \hline 000_2 \end{array}$$

$$\begin{array}{r} 1011_2 \\ +\ 1101_2 \\ \hline 11000_2 \end{array}$$

Notice in the last step we added three ones which total three, and "three" is written as

$$11_2$$

EXAMPLE: Add 111_2, 101_2, and 11_2.
SOLUTION: Write

$$\begin{array}{r} 1 \\ 111_2 \\ 101_2 \\ +\ \ 11_2 \\ \hline 1_2 \end{array}$$

$$\begin{array}{r} 1 \\ 111_2 \\ 101_2 \\ +\ \ 11_2 \\ \hline 11_2 \end{array}$$

$$\begin{array}{r} 111_2 \\ 101_2 \\ +\ \ 11_2 \\ \hline 1111_2 \end{array}$$

We may verify this by writing

$$\begin{array}{rcr} 111_2 & = & 7 \\ 101_2 & = & 5 \\ +\ \ 11_2 & = & 3 \\ \hline & & 15 \end{array}$$

and 15 in base two is 1111_2.

PROBLEMS: Add the following base two numbers.

1. $\begin{array}{r} 11 \\ +\ 101 \\ \hline \end{array}$

2. $\begin{array}{r} 101 \\ +\ 1010 \\ \hline \end{array}$

3. $\begin{array}{r} 11 \\ 11 \\ 11 \\ +\ 11 \\ \hline \end{array}$

ANSWERS:

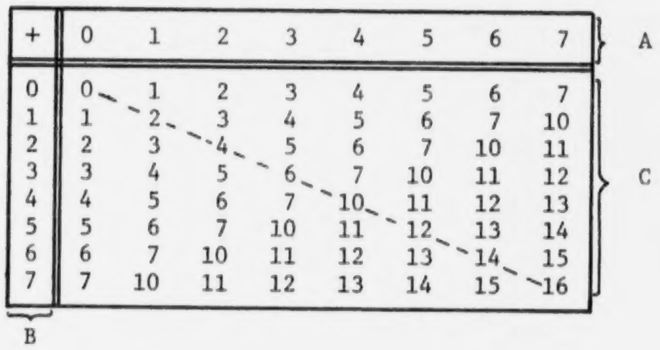
1. 1000_2

2. 1111_2

3. 1100_2

Octal

The octal system has the digits 0, 1, 2, 3, 4, 5, 6, and 7. When an addition carry is made, the carry is eight. The addition facts are shown in figure 5-5.



+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

Figure 5-5.—Octal addition.

When we add 7_8 and 6_8 we have a sum of thirteen but thirteen in base eight is one group of eight and five groups of one. We write

$$\begin{array}{r} 7_8 \\ +\ 6_8 \\ \hline 15_8 \end{array}$$

EXAMPLE: Add 765_8 and 675_8.
SOLUTION: Write

$$\begin{array}{r} 1 \\ 765_8 \\ +\ 675_8 \\ \hline 2_8 \end{array}$$

$$\begin{array}{r} 1 \\ 765_8 \\ +\ 675_8 \\ \hline 62_8 \end{array}$$

$$\begin{array}{r} 765_8 \\ +\ 675_8 \\ \hline 1662_8 \end{array}$$

PROBLEMS: Add the following octal numbers.

1. $\begin{array}{r} 332 \\ +\ 436 \\ \hline \end{array}$

2. $\begin{array}{r} 703 \\ +\ 677 \\ \hline \end{array}$

3. $\begin{array}{r} 4562 \\ +\ 7541 \\ \hline \end{array}$

ANSWERS:

1. 770_8
2. 1602_8
3. 14323_8

Duodecimal

The addition facts for the base twelve system are shown in figure 5-6. The t equals ten and the e equals eleven. When a carry is made the carry is twelve. When we add 9_{12} and e_{12} we find the sum is twenty and twenty is written as one group of twelve and eight groups of one; that is,

$$\begin{array}{r} 9_{12} \\ +\ e_{12} \\ \hline 18_{12} \end{array}$$

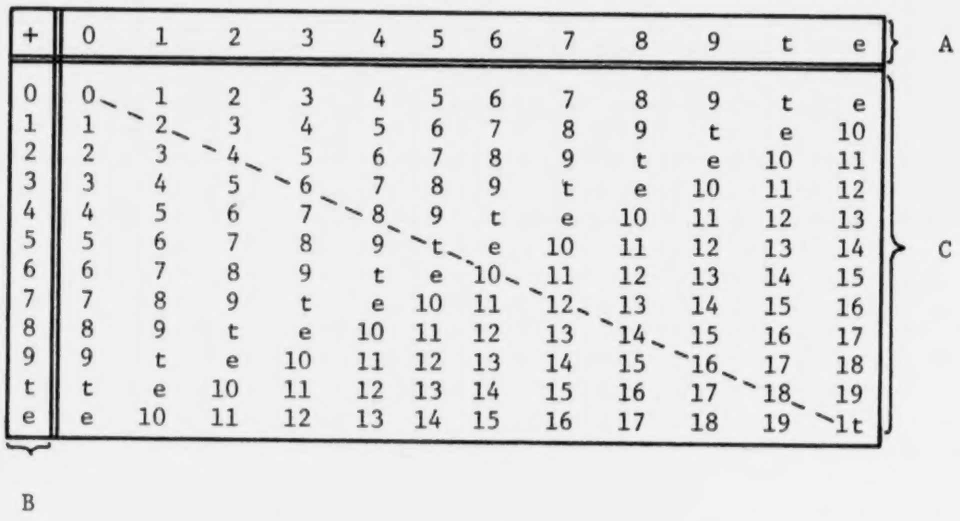
EXAMPLE: Add $8te2_{12}$ and $9e4_{12}$.
SOLUTION: Write

$$\begin{array}{r} 8te2_{12} \\ +\ 9e4_{12} \\ \hline 6_{12} \end{array}$$

$$\begin{array}{r} 1\phantom{2_{12}} \\ 8te2_{12} \\ +\ 9e4_{12} \\ \hline t6_{12} \end{array}$$

$$\begin{array}{r} 1\phantom{e2_{12}} \\ 8te2_{12} \\ +\ 9e4_{12} \\ \hline 8t6_{12} \end{array}$$

$$\begin{array}{r} 8te2_{12} \\ +\ 9e4_{12} \\ \hline 98t6_{12} \end{array}$$



+	0	1	2	3	4	5	6	7	8	9	t	e
0	0	1	2	3	4	5	6	7	8	9	t	e
1	1	2	3	4	5	6	7	8	9	t	e	10
2	2	3	4	5	6	7	8	9	t	e	10	11
3	3	4	5	6	7	8	9	t	e	10	11	12
4	4	5	6	7	8	9	t	e	10	11	12	13
5	5	6	7	8	9	t	e	10	11	12	13	14
6	6	7	8	9	t	e	10	11	12	13	14	15
7	7	8	9	t	e	10	11	12	13	14	15	16
8	8	9	t	e	10	11	12	13	14	15	16	17
9	9	t	e	10	11	12	13	14	15	16	17	18
t	t	e	10	11	12	13	14	15	16	17	18	19
e	e	10	11	12	13	14	15	16	17	18	19	1t

Figure 5-6.—Duodecimal addition.

PROBLEMS: Add the following duodecimal numbers.

1.
$$\begin{array}{r} t \\ +\ e \\ \hline \end{array}$$

2.
$$\begin{array}{r} 9t6 \\ +\ e45 \\ \hline \end{array}$$

3.
$$\begin{array}{r} te7t \\ +\ 9979 \\ \hline \end{array}$$

ANSWERS:

1. 19_{12}
2. $192e_{12}$
3. 18937_{12}

Hexadecimal

The hexadecimal or base sixteen system addition facts are shown in figure 5-7. There are sixteen symbols needed; therefore, the letters A, B, C, D, E, and F are used for digits greater than 9. In addition in this system groups of sixteen are carried.

EXAMPLE: Add $3A9_{16}$ and $E86_{16}$.

SOLUTION: Write

$$\begin{array}{r} 0 \\ 3A9_{16} \\ +\ E86_{16} \\ \hline F_{16} \end{array}$$

+	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
1	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10
2	2	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11
3	3	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12
4	4	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13
5	5	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14
6	6	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15
7	7	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16
8	8	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17
9	9	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18
A	A	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19
B	B	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A
C	C	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B
D	D	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C
E	E	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D
F	F	10	11	12	13	14	15	16	17	18	19	1A	1B	1C	1D	1E

α

γ

β

Figure 5-7.—Hexadecimal addition.

$$\begin{array}{r} 1 \\ 3A9_{16} \\ +\ E86_{16} \\ \hline 2F_{16} \end{array}$$

$$\begin{array}{r} 3A9_{16} \\ +\ E86_{16} \\ \hline 122F_{16} \end{array}$$

EXAMPLE: Add $BC2_{16}$ and EFA_{16}.
SOLUTION: Write

$$\begin{array}{r} 0 \\ BC2_{16} \\ +\ EFA_{16} \\ \hline C_{16} \end{array}$$

$$\begin{array}{r} 1 \\ BC2_{16} \\ +\ EFA_{16} \\ \hline BC_{16} \end{array}$$

$$\begin{array}{r} BC2_{16} \\ +\ EFA_{16} \\ \hline 1ABC_{16} \end{array}$$

PROBLEMS: Add the following hexadecimal numbers.

1. $\begin{array}{r} 9A6 \\ +\ B84 \\ \hline \end{array}$

2. $\begin{array}{r} ABC \\ +\ EF9 \\ \hline \end{array}$

3. $\begin{array}{r} 87A2 \\ +\ F9EC \\ \hline \end{array}$

ANSWERS:

1. $152A_{16}$
2. $19B5_{16}$
3. $1818E_{16}$

SUBTRACTION

Subtraction in any number system is performed in the same manner as in the decimal system. In the process of addition we were faced with the "carry," and in subtraction we are faced with "borrowing."

Since the process of subtraction is the opposite of addition, we may use the addition tables for subtraction facts for the various bases discussed previously.

Decimal

Figure 5-2 is the addition table for the decimal system. Since this table indicates that

$$A + B = C$$

we may use this table for subtraction facts by writing

$$A + B = C$$

then

$$C - A = B$$

or

$$C - B = A$$

To subtract 8 from 15, find 8 in either the A row or B column. Find where this row or column intersects with a value of 15 for C, then move to the remaining row or column to find the remainder.

This problem, when written in the familiar form of

$$\begin{array}{rl} 15 & \text{minuend} \\ -\ 8 & \text{subtrahend} \\ \hline 7 & \text{remainder} \end{array}$$

requires the use of the "borrow"; that is, when we try to subtract 8 from 5 to obtain a positive remainder, we cannot accomplish this. We borrow the 1 which is really one group of ten. Then, one group of ten and 5 groups of one equals 15 groups of one. Then, 15 groups less 8 groups gives the 7 remainder. While this may seem trivial, it nevertheless points out the process used in all number bases. This process may become confusing, when the base is something other than the familiar decimal base.

Quinary

Figure 5-3 may be used in quinary subtraction in the same manner as figure 5-2 was used in decimal subtraction; that is, 4_5 from 12_5 is 3_5. To find this difference locate 4 in the A row, move down to 12_5 in the C array, then across to 3 in the B column. In the familiar form of

$$\begin{array}{r} 12_5 \\ -\ 4_5 \\ \hline 3_5 \end{array}$$

we find we must borrow the 1 from the left place value. This 1 is really one group of 5, therefore one group of 5 added to 2 is 7 and 4 from 7 is 3.

Notice that we solve the previous problem by thinking in base ten but writing in base five. This is permissible because the previous problem may be shown as follows:

$$\begin{aligned} & 12_5 - 4_5 \\ &= 1(5)^1 + 2(5)^0 - 4(5)^0 \\ &= 1(4 + 1) + 2(1) - 4(1) \\ &= 4 + 1 + 2 - 4 \\ &= 4 - 4 + 1 + 2 \\ &= 0 + 1 + 2 \\ &= 3 \end{aligned}$$

where all numbers are in base five.

EXAMPLE: Subtract 43_5 from 431_5.

SOLUTION: Write

$$\begin{array}{r} 431_5 \\ -\ 43_5 \\ \hline \end{array}$$

Thinking in decimal notation, we borrow 1 group of five from the 3 groups of five and write

$$\begin{array}{r} 2\,5 \\ 4\cancel{3}1_5 \\ -\ 43_5 \\ \hline 3_5 \end{array}$$

and then borrow 1 group of $(5)^2$ from the 4 groups of $(5)^2$ which gives

$$\begin{array}{r} 3\,5\,5 \\ \cancel{4}21_5 \\ -\ 43_5 \\ \hline 3_5 \end{array}$$

and we write

$$\begin{array}{r} 5\,5 \\ 321_5 \\ -\ 43_5 \\ \hline 333_5 \end{array}$$

Notice that in the indicated "borrow" we write the numeral which indicates the base. This is for explanation purposes only; there is really no numeral 5 in base five. This holds for the following indicated "borrowing" process in the other bases.

PROBLEMS: Find the remainder in the following.

1. $\begin{array}{r} 23_5 \\ -\ 4_5 \\ \hline \end{array}$

2. $\begin{array}{r} 432_5 \\ -\ 344_5 \\ \hline \end{array}$

3. $\begin{array}{r} 4032_5 \\ -\ 343_5 \\ \hline \end{array}$

ANSWERS:

1. 14_5
2. 33_5
3. 3134_5

Binary

When subtracting in base two, the addition table in figure 5-4 is used. To subtract 1_2 from 10_2 the borrow of two is used. That is,

$$\begin{array}{r} 10_2 \\ -\ 1_2 \\ \hline \end{array}$$

is one group of $(2)^1$ and no group of $(2)^0$ minus one group of $(2)^0$. Thinking in base ten, this is 2 minus 1 which is 1. This may be verified by using figure 5-4.

EXAMPLE: Subtract 11_2 from 101_2.
SOLUTION: Write

$$\begin{array}{r} 101_2 \\ -\ 11_2 \\ \hline \end{array}$$

Then, 1 from 1 is 0 and write

$$\begin{array}{r} 101_2 \\ -\ 11_2 \\ \hline 0_2 \end{array}$$

Now, borrow the left hand 1 which has the value two when moved to the next column to the right. 1 from 2 is 1, and

$$\begin{array}{r} 2 \\ \cancel{1}01_2 \\ -\ 11_2 \\ \hline 10_2 \end{array}$$

PROBLEMS: Perform the indicated operation in the following.

1. $$\begin{array}{r} 11_2 \\ -\ 1_2 \\ \hline \end{array}$$

2. $$\begin{array}{r} 1011_2 \\ -\ 101_2 \\ \hline \end{array}$$

3. $$\begin{array}{r} 1000_2 \\ -\ 101_2 \\ \hline \end{array}$$

ANSWERS:

1. 10_2
2. 110_2
3. 11_2

Octal

Figure 5-5 contains the octal subtraction facts in that

$$C - B = A$$

or

$$C - A = B$$

EXAMPLE: Find the remainder when 6_8 is subtracted from 13_8.
SOLUTION: If, in figure 5-5,

$$C = 13_8$$

and

$$B = 6_8$$

then

$$C - B = A$$

$$13_8 - 6_8 = 5_8$$

EXAMPLE: Subtract 326_8 from 432_8.
SOLUTION: Write

$$\begin{array}{r} 432_8 \\ -\ 326_8 \\ \hline \end{array}$$

then, borrow one group of eight from the 3 which gives eight plus two equals ten. Six from ten is four. Write

$$\begin{array}{r} 28 \\ 4\cancel{3}2_8 \\ -\ 326_8 \\ \hline 4_8 \end{array}$$

and then

$$\begin{array}{r} 8 \\ 422_8 \\ -\ 326_8 \\ \hline 104_8 \end{array}$$

PROBLEMS: Find the remainder in the following.

1. $\begin{array}{r} 637_8 \\ -\ 226_8 \\ \hline \end{array}$

2. $\begin{array}{r} 3206_8 \\ -\ 2737_8 \\ \hline \end{array}$

3. $\begin{array}{r} 4006_8 \\ -\ 1767_8 \\ \hline \end{array}$

ANSWERS:

1. 411_8
2. 247_8
3. 2017_8

Duodecimal

Through the use of figure 5-6 we find that 13_{12} minus 9_{12} is 6_{12}. This may be explained by writing

$$\begin{array}{r} 13_{12} \\ -\ 9_{12} \\ \hline \end{array}$$

We borrow one group of twelve and add it to the three groups of one to obtain fifteen. Then, nine from fifteen is six. Therefore,

$$13_{12} - 9_{12} = 6_{12}$$

Here, as before, we think in base ten and write in the base being used.

EXAMPLE: Subtract $2e9_{12}$ from $t64_{12}$.

SOLUTION: Write

$$\begin{array}{r} t64_{12} \\ -\ 2e9_{12} \\ \hline \end{array}$$

Borrow one group of twelve and add it to four to obtain sixteen. Then nine from sixteen is seven. Write

$$\begin{array}{rrr} & 5 & 12 \\ t & \cancel{6} & 4_{12} \\ -\ 2 & e & 9_{12} \\ \hline & & 7_{12} \end{array}$$

Then, borrow one group from t, the $(12)^2$ column, and add it to the five groups of $(12)^1$ to obtain seventeen groups of $(12)^1$ minus e groups of $(12)^1$ for a remainder of six groups of $(12)^1$. Write

$$\begin{array}{rrr} 9 & 12 & 12 \\ \cancel{t} & 5 & 4_{12} \\ -\ 2 & e & 9_{12} \\ \hline & 6 & 7_{12} \end{array}$$

then, 2_{12} from 9_{12} is 7_{12}, therefore,

$$\begin{array}{rrr} 9 & 12 & 12 \\ \cancel{t} & 5 & 4_{12} \\ -\ 2 & e & 9_{12} \\ \hline 7 & 6 & 7_{12} \end{array}$$

PROBLEMS: Find the remainder in the following.

1. $\begin{array}{r} 96e_{12} \\ -\ 25t_{12} \\ \hline \end{array}$

2. $\begin{array}{r} 6t9_{12} \\ -\ 37e_{12} \\ \hline \end{array}$

3. $\begin{array}{r} e76_{12} \\ -\ 9te_{12} \\ \hline \end{array}$

ANSWERS:

1. 711_{12}
2. $32t_{12}$
3. 187_{12}

Hexadecimal

In figure 5-7 we use the symbols α (Alpha), β (Beta), and γ (Gamma) in place of A, B, and C for the row, column, and array, because A, B, and C are used as symbols in the hexadecimal system. If

$$\alpha + \beta = \gamma$$

then

$$\gamma - \beta = \alpha$$

and

$$\gamma - \alpha = \beta$$

Subtraction in this system is the same as in the other systems previously discussed except a borrow of sixteen is made when required.

EXAMPLE: Find the remainder when $39E_{16}$ is subtracted from $9C6_{16}$.

SOLUTION: Write

$$\begin{array}{r} 9C6_{16} \\ -\ 39E_{16} \\ \hline \end{array}$$

Then, borrow one group of sixteen from C and add it to six to obtain twenty-two. E (fourteen) from twenty-two is eight. Write

$$\begin{array}{rccc} & & B & 16 \\ & 9 & \cancel{C} & 6_{16} \\ - & 3 & 9 & E_{16} \\ \hline & & & 8_{16} \end{array}$$

Then, nine from B (eleven) is two, therefore

$$\begin{array}{rccc} & & & 16 \\ & 9 & B & 6_{16} \\ - & 3 & 9 & E_{16} \\ \hline & & 2 & 8_{16} \end{array}$$

and nine minus three is six, then

$$\begin{array}{rccc} & & & 16 \\ & 9 & B & 6_{16} \\ - & 3 & 9 & E_{16} \\ \hline & 6 & 2 & 8_{16} \end{array}$$

PROBLEMS: Find the remainder in the following.

1. $\begin{array}{r} 6F3_{16} \\ -\ 26A_{16} \\ \hline \end{array}$

2. $\begin{array}{r} 70C_{16} \\ -\ 3FD_{16} \\ \hline \end{array}$

3. $\begin{array}{r} DEF_{16} \\ -\ ABC_{16} \\ \hline \end{array}$

ANSWERS:

1. 489_{16}
2. $30F_{16}$
3. 333_{16}

Subtraction By Complements

Digital computers are generally unable to perform subtraction in the manner previously discussed because the process of borrowing is inconvenient and expensive to mechanize. Therefore, the process of addition of complements is used in place of subtraction. By complement we mean the number or quantity required to fill or complete something in respect to a known reference.

The nines complement of a decimal number is that number which, when added to an original number, will yield nines in each place value column of the original number; that is, the nines complement of 32 is 67 because when 67 is added to 32 the sum is 99.

If we now add one to the nines complement of 32, that is,

$$\begin{array}{r} 67 \\ +\ \ 1 \\ \hline 68 \end{array}$$

we have the tens complement of 32. Notice that if we add a number (32) and its tens complement (68) we have a sum of 100. Therefore, we define the tens complement of a number as that number which, when added to the original number, yields a 1 in the next higher place value column than the highest contained in the original number. This 1 is followed by zeros in all other place value columns. The tens complement of 39 is

$$100 - 39 = 61$$

Notice that the tens complement is in reference to a power of ten equal to the number of place value columns in the original number. This may be shown as follows:

Number	Tens Complement	Reference
8	2	10^1
36	64	10^2
704	296	10^3

Rather than subtract the subtrahend from the minuend we may add the tens complement of the subtrahend (found with reference to the power of ten one place value higher than either the subtrahend or minuend) to the minuend and then decrease this sum by the reference power of ten used. A step-by-step process is shown to explain the preceding statement.

EXAMPLE: Subtract 26 from 49 using complements.

SOLUTION: Write

$$\begin{array}{r} 49 \\ \underline{-\ 26} \end{array}$$

The complement of 26 is 74:

$$100 - 26 = 74$$

This may be rearranged as

$$100 - 74 = 26$$

Now, instead of writing

$$49 - 26$$

write

$$\begin{aligned} &49 - (100 - 74) \\ &= 49 - 100 + 74 \\ &= 49 + 74 - 100 \\ &= 23 \end{aligned}$$

This indicates that the minuend (49) plus the tens complement (74) of the subtrahend are added, then the reference power of ten used (100) is subtracted to give the difference of (23). Notice that we could write

$$\begin{array}{r} 49 \\ \underline{-\ 26} \end{array}$$

equals

$$\begin{array}{l} \quad 49 \\ \underline{+\ \ 74 - 100} \\ \quad 123 - 100 \\ = 23 \end{array}$$

We had only to drop the digit (1) found in the left place value column higher than the highest place value column of the subtrahend. Generally, this digit (1), which is developed, indicates that the difference (23) is a positive value.

EXAMPLE: Subtract 36 from 429 using complements.

SOLUTION: Write

$$\begin{array}{rl} 429 & \text{minuend} \\ \underline{-\ \ 36} & \text{subtrahend} \end{array}$$

then

$$\begin{array}{rl} 429 & \text{minuend} \\ \underline{+\ 964} & \text{tens complement of subtrahend} \\ & \text{with reference to } 10^3 \\ 1393 & \end{array}$$

Now, rather than drop the 1 in 1393 change it to + which indicates a positive answer of 393.

The procedure for subtracting a larger number from a smaller number is slightly different from the previous example. The 1 developed previously will not be developed and a zero will replace the 1. The zero indicates a negative value but the apparent difference is the complement of the true remainder. This is shown in the following example.

EXAMPLE: Subtract 362 from 127.

SOLUTION: Write

$$\begin{array}{rl} 127 & \text{minuend} \\ \underline{-\ 362} & \text{subtrahend} \end{array}$$

then

127 minuend

+ 638 tens complement of subtrahend

0765 apparent remainder

Since there was no 1 generated, we find the true remainder by taking the complement of the apparent remainder and also replacing the 0 with a negative sign to indicate a negative value.

The process of subtraction by using complements in binary is similar to that of decimal. The ones complement of a binary number is found by replacing all ones by zeros and all zeros by ones. This process is called inversion. Therefore, the ones complement of 1001_2 is 0110_2. The twos complement is found by adding a 1 to the ones complement; that is,

Number	Ones Complement	Twos Complement
101_2	010_2	011_2
1011_2	0100_2	0101_2
1111_2	0000_2	0001_2

Notice that the number 101_2 plus its twos complement 011_2 equals 1000_2 which has a 1 developed in the next higher place value column and followed by zeros. The same technique is followed in binary as was used in decimal.

EXAMPLE: Subtract 01101_2 from 11001_2 using complements.

SOLUTION: Write

11001_2 minuend

$-\ 01101_2$ subtrahend

then

11001_2 minuend

$+\ 10011_2$ twos complement of subtrahend

101100_2 difference

$= +\ 01100$ positive difference

As in the decimal system, the 1 which is developed indicates a positive answer. If a zero is present, it indicates a negative apparent answer and the true answer is the complement of the apparent answer.

EXAMPLE: Subtract 1011_2 from 1001_2 using complements.

SOLUTION: Write

1001_2 minuend

$-\ 1011_2$ subtrahend

then

1001_2 minuend

$+\ 0101_2$ twos complement of subtrahend

01110_2 apparent difference

$= -\ 0010_2$ twos complement of the apparent difference with the zero indicating a negative difference

CAUTION: When finding the twos complement of a number, do not forget to add a 1 after the inversion process.

PROBLEMS: Subtract in binary using the complements.

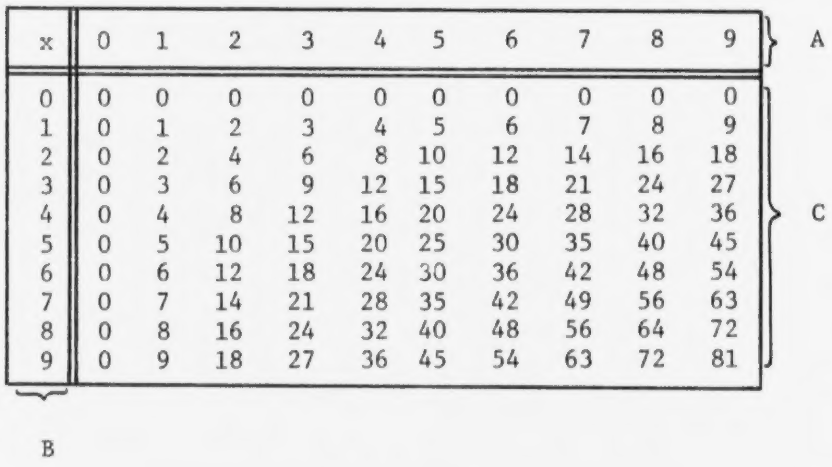
1. 1101 from 1110
2. 101 from 1011
3. 1101 from 1001
4. 111 from 10

ANSWERS:

1. $+\ 0001_2$
2. $+\ 0110_2$
3. $-\ 0100_2$
4. $-\ 101_2$

MULTIPLICATION

Multiplication in any number system is performed in the same manner as in the decimal system. Each system has a unique digit multiplication table. These tables will be discussed with each system. The rows, columns, and arrays of these tables are labeled in the same fashion as the addition tables. Only the sign of operation and array values are different.



x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Figure 5-8.—Decimal multiplication.

Decimal

In multiplication in the decimal system, certain rules are followed which use the decimal digit multiplication and decimal digit addition tables. These rules are well known and apply to direct multiplication in any number system. Figure 5-8 shows the decimal multiplication facts.

The direct method of multiplication of decimal numbers is shown in the following example.

EXAMPLE: Multiply 32 by 25.

SOLUTION: Write

$$25 = 20 + 5$$

then

$$\begin{aligned} &32(25) \\ &= 32(20 + 5) \\ &= 32(20) + 32(5) \\ &= 640 + 160 \\ &= 800 \end{aligned}$$

The same problem written as

$$\begin{array}{r} 32 \\ \underline{\text{x } 25} \end{array}$$

gives

$$\begin{aligned} &32(5) \\ &= 160 \quad \text{partial product} \end{aligned}$$

then

$$\begin{aligned} &32(20) \\ &= 640 \quad \text{partial product} \end{aligned}$$

then

$$\begin{aligned} &160 + 640 \\ &= 800 \quad \text{product} \end{aligned}$$

The technique generally used is

$$\begin{array}{r} 32 \\ \underline{\text{x } 25} \\ 160 \\ \underline{64} \\ 800 \end{array}$$

Notice that the 64 really represents 640 but the zero is omitted.

EXAMPLE: Multiply 306 by 762.

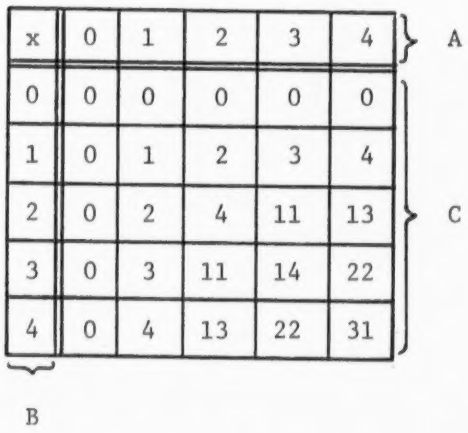
SOLUTION: Write

$$\begin{array}{rl} 306 & \text{factor} \\ \underline{\text{x } 762} & \text{factor} \\ 612 & \text{partial product} \\ 1836 & \text{partial product} \\ \underline{2142} & \text{partial product} \\ 233172 & \text{product} \end{array}$$

Quinary

Figure 5-9 shows the multiplication facts for base five. Notice that when we multiply 3_5 by 2_5 we think in base ten and write the product in base five; that is, $3_5 \times 2_5$ is six, and six in base five is one group of $(5)^1$ and one group of $(5)^0$. Therefore, we write

$$\begin{array}{r} 3_5 \\ \times\ 2_5 \\ \hline 11_5 \end{array}$$



x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

Figure 5-9.—Quinary multiplication.

EXAMPLE: Multiply 304_5 by 24_5.
SOLUTION: Write

$$\begin{array}{r} 304_5 \\ \times\ 24_5 \\ \hline \end{array}$$

Then, four times four is sixteen and sixteen in base five is 31_5. (See fig. 5-9.) Therefore, we write

$$\begin{array}{r} 304_5 \\ \times\ 24_5 \\ \hline 1_5 \end{array}$$

Now, four times zero is zero and the carry of three gives three. Therefore,

$$\begin{array}{r} 304_5 \\ \times\ 24_5 \\ \hline 31_5 \end{array}$$

and $3_5 \times 4_5$ gives 22_5. (See fig. 5-9.) The first partial product is

$$\begin{array}{rl} 304_5 & \\ \times\ 24_5 & \\ \hline 2231_5 & \text{partial product} \end{array}$$

This same procedure is used to find the second partial product as

$$\begin{array}{r} 304_5 \\ \times\ 24_5 \\ \hline 2231_5 \\ 1113_5 \end{array}$$

Then, adding the partial products we find

$$\begin{array}{rl} 304_5 & \\ \times\ 24_5 & \\ \hline 2231_5 & \\ 1113_5 & \\ \hline 13411_5 & \text{product} \end{array}$$

PROBLEMS: Multiply the following:

1. $\begin{array}{r} 23_5 \\ \times\ 41_5 \\ \hline \end{array}$
2. $\begin{array}{r} 3004_5 \\ \times\ 321_5 \\ \hline \end{array}$
3. $\begin{array}{r} 4342_5 \\ \times\ 434_5 \\ \hline \end{array}$

ANSWERS:

1. 2043_5
2. 2020334_5
3. 4233133_5

Binary

Figure 5-10 shows the multiplication facts for the binary system. This is the most simple set of facts of any of the number systems and as will be seen the only difficulty in binary multiplication may be in the addition of the partial products.

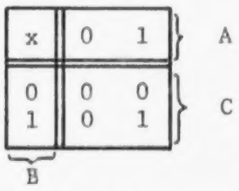


Figure 5-10.—Binary multiplication.

EXAMPLE: Multiply 101_2 by 1101_2.
SOLUTION: Write

$$\begin{array}{r} 1101_2 \\ \underline{\times\ 101_2} \end{array}$$

The partial products and the products are as follows:

$$\begin{array}{rl} 1101_2 & \\ \underline{\times\ 101_2} & \\ 1101 & \text{partial product} \\ \underline{11010} & \text{partial product} \\ 1000001 & \text{product} \end{array}$$

As in the addition section, the problem that may be encountered in the addition of the partial products is what to carry. The following example will illustrate this problem.
EXAMPLE: Multiply 1111_2 by 111_2
SOLUTION: Write

$$\begin{array}{r} 1111_2 \\ \underline{\times\ 111_2} \\ 1111 \\ 1111 \\ 1111 \end{array}$$

We add the partial products by writing

$$\begin{array}{r} 1111_2 \\ \underline{\times\ 111_2} \\ 1111 \\ 1111 \\ \underline{1111} \\ 01 \end{array}$$

and when we add the four ones we find four is written in binary as 100_2. We write the zero, then we must carry the 10_2. The symbol 10_2 is really two, thinking in base ten; therefore, we carry two and when two is added to the next three ones we have five. Five is written as 101_2; therefore, we write 1 and carry the 10_2 or two. Two and two are four so we write zero and carry 10_2 or two. Finally, two and one are three and we write 11_2. The entire addition process is shown as follows:

$$\begin{array}{r} 1111 \\ 1111 \\ \underline{1111} \\ 1101001_2 \end{array}$$

PROBLEMS: Multiply the following.

1. $\begin{array}{r} 1010_2 \\ \underline{\times\ 101_2} \end{array}$

2. $\begin{array}{r} 1110_2 \\ \underline{\times\ 111_2} \end{array}$

3. $\begin{array}{r} 11011_2 \\ \underline{\times\ 1101_2} \end{array}$

ANSWERS:

1. 110010_2
2. 1100010_2
3. 101011111_2

Octal

Base eight multiplication facts are given in figure 5-11. When multiplying 6_8 by 7_8 we find the product by thinking "six times seven is forty-two" and writing forty-two as five groups of eight and two groups of one or

$$6_8 \times 7_8 = 52_8$$

x	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	10	12	14	16
3	0	3	6	11	14	17	22	25
4	0	4	10	14	20	24	30	34
5	0	5	12	17	24	31	36	43
6	0	6	14	22	30	36	44	52
7	0	7	16	25	34	43	52	61

Figure 5-11.—Octal multiplication.

EXAMPLE: Multiply 41_8 by 23_8.
SOLUTION: Write

$$\begin{array}{r} 41_8 \\ \underline{\times\ 23_8} \\ 143 \\ \underline{102} \\ 1163_8 \end{array}$$

PROBLEMS: Multiply the following.

1. $\begin{array}{r} 703_8 \\ \underline{\times\ 24_8} \end{array}$

2. $\begin{array}{r} 324_8 \\ \underline{\times\ 103_8} \end{array}$

3. $\begin{array}{r} 762_8 \\ \underline{\times\ 765_8} \end{array}$

ANSWERS:

1. 21474_8
2. 33574_8
3. 747232_8

Duodecimal

Multiplication facts for the base twelve system are shown in figure 5-12. The process of multiplication is the same as in other bases.

EXAMPLE: Multiply 9_{12} by 5_{12}.
SOLUTION: Write

$$\begin{array}{r} 9_{12} \\ \underline{\times\ 5_{12}} \end{array}$$

x	0	1	2	3	4	5	6	7	8	9	t	e
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	t	e
2	0	2	4	6	8	t	10	12	14	16	18	1t
3	0	3	6	9	10	13	16	19	20	23	26	29
4	0	4	8	10	14	18	20	24	28	30	34	38
5	0	5	t	13	18	21	26	2e	34	39	42	47
6	0	6	10	16	20	26	30	36	40	46	50	56
7	0	7	12	19	24	2e	36	41	48	53	5t	65
8	0	8	14	20	28	34	40	48	54	60	68	74
9	0	9	16	23	30	39	46	53	60	69	76	83
t	0	t	18	26	34	42	50	5t	68	76	84	92
e	0	e	1t	29	38	47	56	65	74	83	92	t1

Figure 5-12.—Duodecimal multiplication.

Nine times five is forty-five in base ten, and forty-five is written as three groups of twelve and nine groups of one, in base twelve; that is,

$$\begin{array}{r} 9_{12} \\ \times\ 5_{12} \\ \hline 39_{12} \end{array}$$

EXAMPLE: Multiply 5_{12} by 7_{12}.
SOLUTION: Write

$$\begin{array}{r} 5_{12} \\ \times\ 7_{12} \\ \hline \end{array}$$

and thirty-five in base ten is written as two groups of twelve and eleven groups of one, in base twelve; therefore,

$$\begin{array}{r} 5_{12} \\ \times\ 7_{12} \\ \hline 2e_{12} \end{array}$$

PROBLEMS: Multiply the following.

1. $\begin{array}{r} 29_{12} \\ \times\ 32_{12} \\ \hline \end{array}$

2. $\begin{array}{r} t7_{12} \\ \times\ 31_{12} \\ \hline \end{array}$

3. $\begin{array}{r} te6_{12} \\ \times\ e_{12} \\ \hline \end{array}$

ANSWERS:

1. 886_{12}
2. 2877_{12}
3. $t066_{12}$

Hexadecimal

Figure 5-13 gives the multiplication facts for base sixteen and figure 5-7 gives the addition facts. By use of both of these tables, the following examples and problems become self-explanatory.

EXAMPLE: Find the product of $6C_{16}$ and 98_{16}.
SOLUTION: Write

$$\begin{array}{r} 6C_{16} \\ \times\ 98_{16} \\ \hline 360\phantom{_{16}} \\ 3CC\phantom{0_{16}} \\ \hline 4020_{16} \end{array}$$

EXAMPLE: Find the product of $3A7_{16}$ and 84_{16}.
SOLUTION: Write

$$\begin{array}{r} 3A7_{16} \\ \times\ 84_{16} \\ \hline E9C\phantom{_{16}} \\ 1D38\phantom{0_{16}} \\ \hline 1E21C_{16} \end{array}$$

PROBLEMS: Find the product of the following.

1. $\begin{array}{r} 67_{16} \\ \times\ 31_{16} \\ \hline \end{array}$

2. $\begin{array}{r} ABC_{16} \\ \times\ 32_{16} \\ \hline \end{array}$

3. $\begin{array}{r} 678_{16} \\ \times\ 302_{16} \\ \hline \end{array}$

ANSWERS:

1. $13B7_{16}$
2. $218B8_{16}$
3. $1374F0_{16}$

DIVISION

The process of division is the opposite of multiplication; therefore, we may use the multiplication tables for the various bases to show division facts. We will define division by writing

x	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	α
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
1	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F	
2	0	2	4	6	8	A	C	E	10	12	14	16	18	1A	1C	1E	
3	0	3	6	9	C	F	12	15	18	1B	1E	21	24	27	2A	2D	
4	0	4	8	C	10	14	18	1C	20	24	28	2C	30	34	38	3C	
5	0	5	A	F	14	19	1D	23	28	2D	32	37	3C	41	46	4B	
6	0	6	C	12	18	1E	24	2A	30	36	3C	42	48	4E	54	5A	γ
7	0	7	E	15	1C	23	2A	31	38	3F	46	4D	54	5B	62	69	
8	0	8	10	18	20	28	30	38	40	48	50	58	60	68	70	78	
9	0	9	12	1B	24	2D	36	3F	48	51	5A	63	6C	75	7E	87	
A	0	A	14	1E	28	32	3C	46	50	5A	64	6E	78	82	8C	96	
B	0	B	16	21	2C	37	42	4D	58	63	6E	79	84	8F	9A	A5	
C	0	C	18	24	30	3C	48	54	60	6C	78	84	90	9C	A8	B4	
D	0	D	1A	27	34	41	4E	5B	68	75	82	8F	9C	A9	B6	C3	
E	0	E	1C	2A	38	46	54	62	70	7E	8C	9A	A8	B6	C4	D2	
F	0	F	1E	2D	3C	4B	5A	69	78	87	96	A5	B4	C3	D2	E1	
β																	

Figure 5-13.—Hexadecimal multiplication.

$$\frac{C}{B} = A \text{ if, and only if, } AB = C,\ B \neq 0$$

We show this by use of figure 5-8. That is, if

$$C = 42$$

and

$$B = 7$$

then

$$A = 6$$

Notice that the value of C is the intersection of the values of A and B.

For the remainder of this section on division we will use examples and problems for the various number bases along with their respective multiplication tables.

Decimal

EXAMPLE: Divide 54 by 9.

$$\begin{array}{r} 6 \\ 9\overline{)54} \\ \underline{54} \\ 0 \end{array}$$

EXAMPLE: Divide 252 by 6.
SOLUTION: Write

$$\begin{array}{r} 42 \\ 6/\overline{252} \\ \underline{24} \\ 12 \\ \underline{12} \\ 0 \end{array}$$

Quinary

EXAMPLE: Divide 22_5 by 4_5.
SOLUTION: Write

$$\begin{array}{r} 3_5 \\ 4/\overline{22} \\ \underline{22} \\ 0 \end{array}$$

EXAMPLE: Divide 2013_5 by 3_5.
SOLUTION: Write

$$\begin{array}{r} 321_5 \\ 3/\overline{2013} \\ \underline{14} \\ 11 \\ \underline{11} \\ 03 \\ \underline{3} \\ 0 \end{array}$$

PROBLEMS: Divide the following.

1. 134_5 by 4_5
2. 2231_5 by 4_5
3. 2131_5 by 3_5

ANSWERS:

1. 21_5
2. 304_5
3. 342_5

Octal

EXAMPLE: Divide 234_8 by 6_8.
SOLUTION: Write

$$\begin{array}{r} 32_8 \\ 6/\overline{234} \\ \underline{22} \\ 14 \\ \underline{14} \\ 0 \end{array}$$

EXAMPLE: Divide 765_8 by 4_8.
SOLUTION: Write

$$\begin{array}{rl} 175_8 & \\ 4/\overline{765} & \\ \underline{4} & \\ 36 & \\ \underline{34} & \\ 25 & \\ \underline{24} & \\ 1 & \text{remainder} \end{array}$$

PROBLEMS: Divide the following.

1. 202_8 by 5_8
2. 1634_8 by 7_8
3. 372_8 by 12_8

ANSWERS:

1. 32_8
2. 204_8
3. 31_8

Binary

EXAMPLE: Divide 1111_2 by 11_2.
SOLUTION: Write

$$\begin{array}{r} 101_2 \\ 11/\overline{1111} \\ \underline{11} \\ 011 \\ \underline{11} \\ 0 \end{array}$$

EXAMPLE: Divide 101_2 by 10_2.
SOLUTION: Write

$$\begin{array}{rl} 10_2 & \\ 10\overline{)101} & \\ \underline{10} & \\ 1 & \text{remainder} \end{array}$$

PROBLEMS: Divide the following

1. 10110_2 by 10_2
2. 1000001_2 by 101_2
3. 100000_2 by 100_2

ANSWERS:

1. 1011_2
2. 1101_2
3. 1000_2

Duodecimal

EXAMPLE: Divide 446_{12} by 6_{12}.
SOLUTION: Write

$$\begin{array}{r} 89_{12} \\ 6\overline{)446} \\ \underline{40} \\ 46 \\ \underline{46} \\ 0 \end{array}$$

EXAMPLE: Divide 417_{12} by 5_{12}.
SOLUTION: Write

$$\begin{array}{r} 9e_{12} \\ 5\overline{)417} \\ \underline{39} \\ 47 \\ \underline{47} \\ 0 \end{array}$$

PROBLEMS: Divide the following.

1. $2e4_{12}$ by 4_{12}
2. $1e23_{12}$ by 3_{12}
3. $19t_{12}$ by 2_{12}

ANSWERS:

1. $8t_{12}$
2. 789_{12}
3. te_{12}

Hexadecimal

EXAMPLE: Divide $D4E_{16}$ by 2_{16}.
SOLUTION: Write

$$\begin{array}{r} 6A7_{16} \\ 2\overline{)D4E} \\ \underline{C} \\ 14 \\ \underline{14} \\ E \\ \underline{E} \\ 0 \end{array}$$

EXAMPLE: Divide $13BC_{16}$ by 3_{16}.
SOLUTION: Write

$$\begin{array}{r} 694_{16} \\ 3\overline{)13BC} \\ \underline{12} \\ 1B \\ \underline{1B} \\ C \\ \underline{C} \\ 0 \end{array}$$

PROBLEMS: Divide the following

1. $27B5_{16}$ by 5_{16}
2. $4B24_{16}$ by 7_{16}
3. $2CCE_{16}$ by A_{16}

ANSWERS:

1. $7F1_{16}$
2. ABC_{16}
3. $47B_{16}$

CONVERSIONS

It has been shown that place value is the determining factor in evaluating a number. We will make extensive use of this idea in discussing the various methods of conversions.

NON-DECIMAL TO DECIMAL

In order to convert a non-decimal number to a decimal number we make use of the polynomial form. That is, we write the non-decimal number in polynomial form and then carry out the indicated operations.

EXAMPLE: Convert 634_8 to decimal.
SOLUTION: Write

$$\begin{aligned} 634_8 &= 6(8)^2 + 3(8)^1 + 4(8)^0 \\ &= 6(64) + 3(8) + 4(1) \\ &= 384 + 24 + 4 \\ &= 412 \end{aligned}$$

EXAMPLE: Convert $7t0e_{12}$ to decimal; that is, if

$$7t0e_{12} = X_{10}, \text{ then } X_{10} = ?$$

SOLUTION: Write

$$\begin{aligned} 7t0e_{12} &= 7(12)^3 + 10(12)^2 = 0(12)^1 = 11(12)^0 \\ &= 7(1728) + 10(144) + 0(12) + 11(1) \\ &= 12096 + 1440 + 0 + 11 \\ &= 13547 \end{aligned}$$

Therefore, $X_{10} = 13547$.

Another method of non-decimal to decimal conversion is by synthetic substitution. This method is shown in the following example.

EXAMPLE: Convert 634_8 to decimal.
SOLUTION: Write

$$\begin{array}{r|rrr} 8 & 6 & 3 & 4 \\ \hline \end{array}$$

Bring down the six

$$\begin{array}{r|rrr} 8 & 6 & 3 & 4 \\ \hline & 6 & & \end{array}$$

Multiply the six by the base (expressed in decimal form) and carry the decimal product to the next lower place value column.

$$\begin{array}{r|rrr} 8 & 6 & 3 & 4 \\ & & 48 & \\ \hline & 6 & & \end{array}$$

Add the three and the carried product

$$\begin{array}{r|rrr} 8 & 6 & 3 & 4 \\ & & 48 & \\ \hline & 6 & 51 & \end{array}$$

Multiply this sum by the base and carry to the next lower place value column.

$$\begin{array}{r|rrr} 8 & 6 & 3 & 4 \\ & & 48 & 408 \\ \hline & 6 & 51 & \end{array}$$

Add the four and the carried product to find the decimal equivalent of 634_8 to be

$$\begin{array}{r|rrr} 8 & 6 & 3 & 4 \\ & & 48 & 408 \\ \hline & 6 & 51 & 412 \end{array}$$

and 412 is the decimal equivalent of 634_8. The entire previous process may be shown, without carrying out the multiplication, as

$$\begin{array}{r|lll} 8 & 6 & 3 & 4 \\ \hline & 6 & & \end{array}$$

$$\begin{array}{r|lll} 8 & 6 & 3 & 4 \\ & & 6(8) & \\ \hline & 6 & & \end{array}$$

$$\begin{array}{r|lll} 8 & 6 & 3 & 4 \\ & & 6(8) & \\ \hline & 6 & 6(8) + 3 & \end{array}$$

$$\begin{array}{r|lll} 8 & 6 & 3 & 4 \\ & & 6(8) & 8[6(8)+3] \\ \hline & & 6(8) + 3 & \end{array}$$

$$\begin{array}{r|lll} 8 & 6 & 3 & 4 \\ & & 6(8) & 8[6(8)+3] \\ \hline & 6 & 6(8) + 3 & 8[6(8)+3] + 4 \end{array}$$

and

$$8[6(8)+3] + 4$$

is

$$6(8)^2 + 3(8) + 4$$

which is really 634_8 written in polynomial form.

EXAMPLE: Convert $7te_{12}$ to decimal.

SOLUTION: Write

$$\begin{array}{r|lll} 12 & 7 & 10 & 11 \\ & & 7(12) & 12[7(12)+10] \\ \hline & 7 & 7(12) + 10 & 12[7(12)+10] + 11 \end{array}$$

and

$$\begin{aligned} & 12[7(12) + 10] + 11 \\ &= 7(12)^2 + 10(12)^1 + 11(12)^0 \\ &= 1008 + 120 + 11 \\ &= 1139 \end{aligned}$$

A third method of converting a number from non-decimal to decimal is by use of repeated division where the remainders indicate the decimal equivalent. The denominator is ten expressed in the non-decimal number.

EXAMPLE: Convert 634_8 to decimal.

SOLUTION: Ten expressed in base eight is 12; therefore, write

$$12/\overline{634}$$

This division is carried out in base eight.

$$\begin{array}{r} 51 \\ 12/\overline{634} \\ \underline{62} \\ 14 \\ 12 \\ \hline R_1 = 2_8 = 2 \end{array}$$

The dividend is now divided by 12_8.

$$\begin{array}{r} 4 \\ 12/\overline{51} \\ 50 \\ \hline R_2 = 1_8 = 1 \end{array}$$

This process is continued until the dividend is zero.

$$\begin{array}{r} 0 \\ 12/\overline{4} \\ 0 \\ \hline R_3 = 4_8 = 4 \end{array}$$

Now, if $634_8 = X_{10}$ then

$$X_{10} = R_3 \quad R_2 \quad R_1$$

where

$$R_1 = 2$$

$$R_2 = 1$$

$$R_3 = 4$$

Therefore

$$X_{10} = 412$$

EXAMPLE: Convert $7t0e_{12}$ to decimal.

SOLUTION: Ten expressed in base twelve is t, therefore, write

$$t\overline{)7t0e}$$

This division is carried out in base twelve.

$$\begin{array}{r} 94t \\ t\overline{)7t0e} \\ \underline{76} \\ 40 \\ \underline{34} \\ 8e \\ \underline{84} \\ R_1 = 7_{12} = 7 \end{array}$$

Divide the quotient as follows:

$$\begin{array}{r} e3 \\ t\overline{)94t} \\ \underline{92} \\ 2t \\ \underline{26} \\ R_2 = 4_{12} = 4 \end{array}$$

Then, the quotient e3 is divided by t. That is,

$$\begin{array}{r} 11 \\ t\overline{)e3} \\ \underline{t} \\ 13 \\ \underline{t} \\ R_3 = 5_{12} = 5 \end{array}$$

Further division produces

$$\begin{array}{r} 1 \\ t\overline{)11} \\ \underline{t} \\ R_4 = 3_{12} = 3 \end{array}$$

Then,

$$\begin{array}{r} 0 \\ t\overline{)1} \\ \underline{0} \\ R_5 = 1_{12} = 1 \end{array}$$

Therefore, if $7t0e_{12} = X_{10}$ and

$$X_{10} = R_5 \quad R_4 \quad R_3 \quad R_2 \quad R_1$$

where

$$R_1 = 7$$

$$R_2 = 4$$

$$R_3 = 5$$

$$R_4 = 3$$

$$R_5 = 1$$

then

$$X_{10} = 13547$$

PROBLEMS: Convert the following non-decimal numbers to decimal using each of the three methods discussed.

1. 342_8
2. 431_5
3. $6AC_{16}$

ANSWERS:

1. 226
2. 116
3. 1697

DECIMAL TO NON-DECIMAL

To convert a number from decimal to non-decimal the process of repeated division is used and the remainders indicate the non-decimal number. The denominator is the non-decimal base expressed in base ten and the division process is in base ten.

EXAMPLE: Convert 319 to octal; that is, if

$$319 = X_8, \text{ then } X_8 = ?$$

SOLUTION: Base eight expressed in decimal is 8, therefore, write

$$\begin{array}{r} 39 \\ 8/\overline{319} \\ \underline{24} \\ 79 \\ \underline{72} \\ R_1 = 7 = 7_8 \end{array}$$

and

$$\begin{array}{r} 4 \\ 8/\overline{39} \\ \underline{32} \\ R_2 = 7 = 7_8 \end{array}$$

then

$$\begin{array}{r} 0 \\ 8/\overline{4} \\ \underline{0} \\ R_3 = 4 = 4_8 \end{array}$$

Now, if

$$319 = X_8$$

then

$$X_8 = R_3 \quad R_2 \quad R_1$$

In this example

$$R_1 = 7$$

$$R_2 = 7$$

$$R_3 = 4$$

therefore,

$$X_8 = 477_8$$

EXAMPLE: Convert 18 to binary; that is, if

$$18 = X_2, \text{ then } X_2 = ?$$

SOLUTION: Base two in decimal is 2, therefore, write

$$\begin{array}{r} 9 \\ 2/\overline{18} \\ \underline{18} \\ R_1 = 0 = 0_2 \end{array}$$

and

$$\begin{array}{r} 4 \\ 2/\overline{9} \\ \underline{8} \\ R_2 = 1 = 1_2 \end{array}$$

and

$$\begin{array}{r} 2 \\ 2/\overline{4} \\ \underline{4} \\ R_3 = 0 = 0_2 \end{array}$$

and

$$\begin{array}{r} 1 \\ 2/\overline{2} \\ \underline{2} \\ R_4 = 0 = 0_2 \end{array}$$

then

$$\begin{array}{r} 0 \\ 2/\overline{1} \\ \underline{0} \\ R_5 = 1 = 1_2 \end{array}$$

therefore, if 18 = X_2 and

$$X_2 = R_5 \quad R_4 \quad R_3 \quad R_2 \quad R_1$$

where

$$R_1 = 0$$

$$R_2 = 1$$

$$R_3 = 0$$

$$R_4 = 0$$

$$R_5 = 1$$

then

$$X_2 = 10010_2$$

EXAMPLE: Convert 632 to duodecimal. That is, if

$$632 = X_{12}, \text{ then } X_{12} = ?$$

SOLUTION: Base twelve expressed in base ten is 12. Write

$$\begin{array}{r} 52 \\ 12\overline{)632} \\ \underline{60} \\ 32 \\ \underline{24} \\ R_1 = 8 = 8_{12} \end{array}$$

and

$$\begin{array}{r} 4 \\ 12\overline{)52} \\ \underline{48} \\ R_2 = 4 = 4_{12} \end{array}$$

and

$$\begin{array}{r} 0 \\ 12\overline{)4} \\ \underline{0} \\ R_3 = 4 = 4_{12} \end{array}$$

If

$$632 = X_{12}$$

and

$$X_{12} = R_3 \quad R_2 \quad R_1$$

then

$$X_{12} = 448_{12}$$

EXAMPLE: Convert 128 to duodecimal.
SOLUTION: Write

$$\begin{array}{r} 10 \\ 12\overline{)128} \\ \underline{12} \\ R_1 = 8 = 8_{12} \end{array}$$

and

$$\begin{array}{r} 0 \\ 12\overline{)10} \\ \underline{00} \\ R_2 = 10 = t_{12} \end{array}$$

therefore,

$$128 = t8_{12}$$

The format for the repeated division process may be simplified in cases where the actual division is simple. This is shown in the following example.

EXAMPLE: Convert 34 to binary.

SOLUTION: Carry out the repeated division indicating the remainder to the right of the division; that is,

$$\begin{array}{rcc} & \text{Remainder} & \\ 2\,/\,18 & & \\ 2\,/\,\underline{9} & 0 & \uparrow \\ 2\,/\,\underline{4} & 1 & | \\ 2\,/\,\underline{2} & 0 & | \\ 2\,/\,\underline{1} & 0 & | \\ \underline{0} & 1 & | \end{array}$$

Now read the remainder from the bottom to the top to find the binary equivalent of the decimal number. In this case the binary number is 10010_2.

PROBLEMS: Convert the following decimal numbers to the base indicated.

1. 27 to base two.
2. 123 to base five.
3. 467 to base twelve.
4. 996 to base sixteen.

ANSWERS:

1. 11011_2
2. 443_5
3. $32e_{12}$
4. $3E4_{16}$

NON-DECIMAL TO NON-DECIMAL

We will consider three approaches to the non-decimal to non-decimal conversions. One method will be through base ten and the other two methods will be direct.

When going through base ten, the polynomial form is used along with repeated division.

EXAMPLE: Convert 2143_5 to base eight.

SOLUTION: Convert 2143_5 to base ten by writing this number in polynomial form. That is,

$$\begin{aligned} 2143_5 &= 2(5)^3 + 1(5)^2 + 4(5)^1 + 3(5)^0 \\ &= 2(125) + 1(25) + 4(5) + 3(1) \\ &= 250 + 25 + 20 + 3 \\ &= 298 \end{aligned}$$

We now convert 298 from base ten to base eight by repeated division. That is, write

	Remainder
8/298	
8/37	2
8/4	5
0	4

therefore,

$$2143_5 = 452_8$$

EXAMPLE: Convert 10110_2 to base twelve.

SOLUTION: Write (polynomial form)

$$\begin{aligned} 10110_2 &= 1(2)^4 + 0(2)^3 + 1(2)^2 + 1(2)^1 + 0(2)^0 \\ &= 16 + 0 + 4 + 2 + 0 \\ &= 22 \end{aligned}$$

Then (repeated division),

	Remainder
12/22	
12/1	t
0	1

therefore,

$$10110_2 = 1t_{12}$$

EXAMPLE: Convert $3C7_{16}$ to base five.

SOLUTION: Write (polynomial form)

$$\begin{aligned} 3C7_{16} &= 3(16)^2 + C(16)^1 + 7(16)^0 \\ &= 768 + 192 + 7 \\ &= 967 \end{aligned}$$

Then (repeated division),

	Remainder
5/967	
5/193	2
5/38	3
5/7	3
5/1	2
0	1

therefore,

$$3C7_{16} = 12332_5$$

The second method of converting a non-decimal number to a non-decimal number is by division. The division is carried out by dividing by the base wanted, performing the calculation in the base given.

EXAMPLE: Convert 2143_5 to base eight.

SOLUTION: The base given is five and the base wanted is eight. Therefore, express eight in base five, obtaining 13_5. We carry out the division by 13_5 in base five as follows:

$$\begin{array}{r} 122 \\ 13\overline{)2143} \\ \underline{13} \\ 34 \\ \underline{31} \\ 33 \\ \underline{31} \\ R_1 = 2_5 = 2_8 \end{array}$$

then

$$\begin{array}{r} 4 \\ 13\overline{)122} \\ \underline{112} \\ R_2 = 10_5 = 5_8 \end{array}$$

and

$$\begin{array}{r} 0 \\ 13\overline{)4} \\ \underline{0} \\ R_3 = 4_5 = 4_8 \end{array}$$

and

$$X_8 = R_3 \quad R_2 \quad R_1$$

then

$$\begin{aligned} 2143_5 &= X_8 \\ &= R_3 \quad R_2 \quad R_1 \\ &= 452_8 \end{aligned}$$

EXAMPLE: Convert $7e6_{12}$ to base five.

SOLUTION: The base given is twelve and the base wanted is five. Therefore, five in base twelve is 5_{12}. The division is carried out in base twelve. Write

$$\begin{array}{r} 171 \\ 5\overline{)7e6} \\ \underline{5} \\ 2e \\ \underline{2e} \\ 6 \\ \underline{5} \\ R_1 = 1_{12} = 1_5 \end{array}$$

then

$$\begin{array}{r} 39 \\ 5\overline{)171} \\ \underline{13} \\ 41 \\ \underline{39} \\ R_2 = 2_{12} = 2_5 \end{array}$$

and

$$\begin{array}{r} 9 \\ 5\overline{)39} \\ \underline{39} \\ R_3 = 0_{12} = 0_5 \end{array}$$

and

$$\begin{array}{r} 1 \\ 5\overline{)9} \\ \underline{5} \\ R_4 = 4_{12} = 4_5 \end{array}$$

then

$$\begin{array}{r} 0 \\ 5\overline{)1} \\ \underline{0} \\ R_4 = 1_{12} = 1_5 \end{array}$$

therefore,

$$7e6_{12} = 14021_5$$

PROBLEMS: Convert the following without going through base ten.

1. 342_5 to base two.
2. $t73_{12}$ to base eight.
3. $A62_{16}$ to base five.

ANSWERS:

1. 1100001_2
2. 2767_8
3. 41113_5

The last method we will discuss is called the explosion method. It consists of the following rules:

1. Perform all arithmetic operations in the desired base.
2. Express the base of the original number in terms of the base of the desired number.
3. Multiply the number obtained in step 2 by the leftmost digit and add the product to the next digit on the right of the original number.

(NOTE: It may be necessary to convert each digit of the original number to an expression conforming to the desired base.)

4. Repeat step 3 as many times as there are digits. The final sum is the answer.

EXAMPLE: Convert 2143_5 to base eight.
SOLUTION: Write

five in base five = 5_8 (Rule 2)

$$\begin{array}{rrrr} (2_8 & 1_8 & 4_8 & 3_8)_5 \\ \underline{\times\ 5_8} & & & \\ 12_8 & & & \end{array}$$

(Rule 3)

then

$$\begin{array}{rrrr} (2_8 & 1_8 & 4_8 & 3_8)_5 \\ \underline{\times\ 5_8} & \underline{+\ 12_8} & & \\ 12_8 & 13_8 & & \\ & \underline{\times\ 5_8} & & \\ & 67_8 & & \end{array}$$

and

$$\begin{array}{rrrr} (2_8 & 1_8 & 4_8 & 3_8)_5 \\ \underline{\times\ 5_8} & \underline{+\ 12_8} & \underline{+\ 67_8} & \\ 12_8 & 13_8 & 73_8 & \\ & \underline{\times\ 5_8} & \underline{\times\ 5_8} & \\ & 67_8 & 447_8 & \end{array}$$

then

$$\begin{array}{rrrr} (2_8 & 1_8 & 4_8 & 3_8)_5 \\ \underline{\times\ 5_8} & \underline{+\ 12_8} & \underline{+\ 67_8} & \underline{+\ 447_8} \\ 12_8 & 13_8 & 73_8 & 452_8 \\ & \underline{\times\ 5_8} & \underline{\times\ 5_8} & \\ & 67_8 & 447_8 & \end{array}$$

Therefore,

$$2143_5 = 452_8$$

EXAMPLE: Convert 452_8 to base five.
SOLUTION: Eight expressed in base five is 13_5. 452 expressed in base five, digit by digit, is 4 10 2. Then,

$$\begin{array}{rrr} 4 & 10 & 2 \\ \underline{\times\ 13_5} & \underline{+\ 112} & \underline{+\ 2141} \\ 22 & 122 & 2143_5 \\ \underline{4} & \underline{\times\ 13} & \\ 112 & 421 & \\ & \underline{122} & \\ & 2141 & \end{array}$$

Therefore,

$$452_8 = 2143_5$$

EXAMPLE: Convert $t62_{12}$ to base five.
SOLUTION: Twelve expressed in base five is 22_5. t62 expressed in base five, digit by digit, is 20 11 2. Then,

$$\begin{array}{rrr} 20 & 11 & 2 \\ \underline{\times\ 22_5} & \underline{+\ 440} & \underline{+\ 22022} \\ 40 & 1001 & 22024_5 \\ \underline{40} & \underline{\times\ 22} & \\ 440 & 2002 & \\ & \underline{2002} & \\ & 22022 & \end{array}$$

therefore,

$$t62_{12} = 22024_5$$

PROBLEMS: Convert the following to the base indicated.

1. 32_5 to base two.
2. 347_8 to base twelve.
3. $te3_{12}$ to base five.

ANSWERS:

1. 10001_2

2. 173_{12}

3. 22300_5

SPECIAL CASES OF CONVERSIONS

Changing from base two to base eight is accomplished rather easily because eight is a power of two. That is, 8 equals 2^3. We need only group our base two number in groups of three digits (the power of the original which gives the new base) and use each group of three digits as a single place value of base eight.

EXAMPLE: Convert 1011001_2 to base eight.

SOLUTION: Group 1011001_2 in groups of three starting at the right. That is,

$$001 \quad 011 \quad 001_{\ 2}$$

then write each group of three digits in base eight.

$$1 \quad 3 \quad 1_{\ 8}$$

Verification may be made by writing

$$1011001_2 = 89$$

and

$$131_8 = 89$$

therefore,

$$1011001_2 = 131_8$$

EXAMPLE: Convert 1001101_2 to base sixteen.

SOLUTION: Sixteen is the fourth power of two so we use groups of four digits. Write

$$0100 \quad 1101_{\ 2}$$

Then write each group of digits in base sixteen.

$$4 \quad D_{\ 16}$$

EXAMPLE: Convert 11011011111_2 to base sixteen.

SOLUTION: Write

$$0110 \quad 1101 \quad 1111_{\ 2}$$

$$= 6 \quad D \quad F_{\ 16}$$

To reverse this process, that is, to convert from base eight to base two, we use three digits in base two to express each digit in base eight. This is because two is the third root of eight.

EXAMPLE: Convert 132_8 to base two.

SOLUTION: Write

$$1 \quad 3 \quad 2_{\ 8}$$

Then write each base eight digit in base two using three digits. That is,

$$1 \quad 3 \quad 2_{\ 8}$$

$$= 001 \quad 011 \quad 010_{\ 2}$$

EXAMPLE: Convert $6A7_{16}$ to base two.

SOLUTION: Two is the fourth root of sixteen; therefore, we express each base sixteen digit in base two using four digits. That is,

$$6 \quad A \quad 7_{\ 16}$$

$$= 0110 \quad 1010 \quad 0111_{\ 2}$$

This process may be used when one base is a power or root of the other base.

PROBLEMS: Convert the following to the base indicated.

1. 11011101_2 to base eight.

2. 4762_8 to base two.

3. $9E7_{16}$ to base two.

4. 111110111_2 to base sixteen.

ANSWERS:

1. 335_8

2. 100111110010_2

3. 100110110111_2

4. $1F7_{16}$

DECIMAL TO BINARY CODED DECIMAL

Although the Binary Coded Decimal (BCD) is not truly a number system, we will discuss this code because it is computer related as some of the bases are.

This code, sometimes called the 8421 code, makes use of groups of binary symbols to represent a decimal number. In the decimal system there are only ten symbols; therefore, only ten groups of binary bits (symbols) must be remembered. Each decimal digit is represented by a group of four binary bits. The ten groups to remember are as follows:

Decimal Symbol	Binary Coded Decimal (BCD)
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001

To express a decimal number as a BCD we use a binary group for each decimal symbol we have; that is,

Decimal 72
BCD 0111 0010

Notice that for every decimal digit we must have one group of binary bits. Thus,

Decimal 3 8 1
BCD 0011 1000 0001

The separation of the BCD groups is shown for ease of reading and does not necessarily need to be written as shown. The number 381 could be written as 001110000001. One advantage of the BCD over true binary is ease of determining the decimal value. This is shown as follows:

Decimal 9 3 4
BCD 1001 0011 0100

The number 934 in true binary is 1110100110_2. This in polynomial form is

$$1(2)^9 + 1(2)^8 + 1(2)^7 + 0(2)^6 + 1(2)^5 + 0(2)^4 + 0(2)^3 + 1(2)^2 + 1(2)^1 + 0(2)^0$$

$$= 512 + 256 + 128 + 0 + 32 + 0 + 0 + 4 + 2 + 0$$

$$= 934$$

In order to change a decimal to BCD we need only write one group of binary bits to represent each decimal digit; that is,

Decimal 7203

is

BCD 0111 0010 0000 0011

To convert a BCD to decimal we group the binary bits in groups of four, from the right, and then write the decimal digit represented by each group. Thus,

BCD 0100100110011000
= BCD 0100 1001 1001 1000
= Decimal 4 9 9 8
= 4998

The ease of converting from decimal to BCD and from BCD back to decimal should be apparent from the following problems.

PROBLEMS: Convert the following decimal numbers to BCD.

1. 6
2. 31
3. 764
4. 3098

ANSWERS:

1. 0110
2. 0011 0001 or 00110001

3. 0111 0110 0100 or 011101100100

4. 0011 0000 1001 1000 or 0011000010011000

PROBLEMS: Convert the following BCD's to decimals.

1. 0010

2. 10011000

3. 011000110111

4. 0101000001111000

ANSWERS:

1. 2

2. 98

3. 637

4. 5078

The comparative ease of conversion in BCD is related to the difficulty of conversion in true binary by the following problems.

PROBLEMS: Convert as follows:

1. 438 to binary

2. 100101101_2 to decimal

ANSWERS:

1. 110110110_2

2. 301

One serious disadvantage of the BCD is that this code cannot provide a "decimal" carry. The following examples are given to show this.

EXAMPLE: Add the following:

Decimal		BCD
5	=	0101
+ 3	=	+ 0011
8	=	1000

Notice that we did not have a carry in the decimal addition and the answer in BCD is equal to the answer in decimal. The BCD is in correct notation and does exist.

EXAMPLE: Add the following:

Decimal		BCD
8	=	1000
+ 5	=	+ 0101
13	=	1101

Notice that the BCD symbol is the true binary representation of 13 but 1101 does not exist in BCD. The correct BCD answer for 13 is 0001 0011. When a carry is made in decimal the BCD system cannot indicate the correct answer in BCD form.

EXCESS THREE CODE

The excess three code is used to eliminate the inability of the decimal carry. It is really a modification of the BCD so that a carry can be made.

To change a BCD symbol to excess three add three to the BCD; that is,

BCD 1000
+ 0011
excess three 1011

The excess three number 1011 is 8 in decimal. The following shows the correspondence between decimal, BCD, and excess three code.

Decimal	BCD	Excess Three
0	0000	0011
1	0001	0100
2	0010	0101
3	0011	0110
4	0100	0111
5	0101	1000
6	0110	1001
7	0111	1010
8	1000	1011
9	1001	1100

As previously stated, the excess three code will provide the capability of the decimal carry. The following is given for explanation.

EXAMPLE: Add 6 and 3 in excess three.
SOLUTION: Write

$$\begin{array}{rcl} \text{Decimal} & & \\ 6 & = & 0011\quad 0011\quad 1001 \\ \underline{+\ 3} & = & \underline{0011\quad 0011\quad 0110} \\ 9 & & \end{array}$$

Notice that in the right-hand groups the six and three are given. In the other groups a zero (0011) is indicated.

Then,

$$\begin{array}{rrrl} 0011 & 0011 & 1001 & \text{(excess three)} \\ +\ 0011 & 0011 & 0110 & \text{(excess three)} \\ \hline 0110 & 0110 & 1111 & \text{(excess six)} \end{array}$$

Our answer is in excess six; therefore, we must subtract three from each group in order to return our answer to excess three; that is,

$$\begin{array}{rrrl} 0110 & 0110 & 1111 & \\ -\ 0011 & 0011 & 0011 & \\ \hline 0011 & 0011 & 1100 & \text{(excess three)} \\ 0 & 0 & 9 & \text{in decimal} \end{array}$$

When a carry is developed in any group, the following procedure is used.

EXAMPLE: Add 9 and 3 in excess three.
SOLUTION: Write

$$\begin{array}{rrrrl} \text{Decimal} & & \text{Excess Three} & & \\ 9 & 0011 & 0011 & 1100 & \text{(excess three)} \\ & & & 1 & \\ +\ 3 & 0011 & 0011 & 0110 & \text{(excess three)} \\ \hline 12 & 0110 & 0111 & 0010 & \text{(excess six)} \end{array}$$

NOTE: Since the right-hand group created a carry, as shown, three must be ADDED instead of subtracted in order to place this group into excess three. The other groups follow the previous example; that is,

$$\begin{array}{rrrl} 0110 & 0111 & 0010 & \text{(excess six)} \\ -\ 0011 & -0011 & +0011 & \\ \hline 0011 & 0100 & 0101 & \text{(excess three)} \\ 0 & 1 & 2 & \text{decimal} \end{array}$$

PROBLEMS: Add the following in excess three.

1. 5 and 3.
2. 9 and 8.
3. 22 and 56.
4. 58 and 77.

ANSWERS:

1. 0011 0011 1011
2. 0011 0100 1010
3. 0011 1010 1011
4. 0100 0110 1000

A further advantage of the excess three code is the ease with which the nines complement of a number indicated in excess three may be found. That is, the nines complement of seven, indicated in excess three as 1010, is found by inverting each digit in 1010 to read 0101. This 0101, in excess three represents decimal two which is the nines complement of seven.

The following shows the nines complement of the decimal digits.

Decimal	Excess Three	Excess Three Nines Complement	Decimal Nines Complement
0	0011	1100	9
1	0100	1011	8
2	0101	1010	7
3	0110	1001	6
4	0111	1000	5
5	1000	0111	4
6	1001	0110	3
7	1010	0101	2
8	1011	0100	1
9	1100	0011	0

CHAPTER 6

SETS AND SUBSETS

Since sets, subsets, and Boolean algebra satisfy the same laws (that is, have similar properties), we will discuss sets and subsets as a means of introducing Boolean algebra. It should be understood, though, that sets and subsets are related to all other branches of mathematics.

SETS

The meaning of a set is any well-defined collection, group, list, or class of objects which possess (or do not possess) some common property whereby we can determine membership in that set. The object or element may take any form such as articles, people, conditions, or numbers. Membership is the fundamental relation of set theory.

NOTATION

When appropriate we will use capital letters to designate sets and lower case letters to indicate elements of a set. Generally, sets are designated by the following:

1. Description.
2. Tabulation.
3. Capital letters.
4. Set builder notation.
5. Combinations of the above.

Examples of each form of notation, in order, are as follows:

1. "the set of odd prime numbers less than ten."
2. $\{7, 3, 5\}$ or $\{0, 2, 4, 6, \cdots\}$
3. A, B, C, $\cdots$
4. $\{x | x$ is a natural number$\}$ or $\{x | x = N\}$
5. $A = \{x | x^2 = 4\}$

In item 4 we let $|$ mean "such that" and x represents any element of the set. Also, the x to the left of $|$ is a variable and the defining property which is required to belong to the set is to the right of $|$.

Item 5 is read as "A is the set of numbers x such that x square equals four."

Examples of sets and their designations are as follows:

1. The numbers 3, 5, 7, and 11.
   $\{3, 5, 7, 11\}$
2. The letters of the alphabet between c and i.
   $\{d, e, f, g, h\}$
3. Members of the Navy.
   $\{x | x$ is a member of the Navy$\}$
4. Solutions of the equation $x^2 + 3x - 4 = 0$.
   $\{x | x^2 + 3x - 4 = 0\}$

If set A contains x as one of its elements, we indicate this membership by writing

$$x \in A$$

and if set A does not contain x, we write

$$x \notin A$$

That is, if

$$A = \{2, 4, 6, 8, \cdots\}$$

then

$$2 \in A,\ 3 \notin A,\ 13 \notin A,\ 12 \in A, \text{ etc.}$$

FINITE AND INFINITE SETS

We define a finite set as one in which its elements or members could be counted; that is, the counting process or enumeration of its elements would at some time come to an end. This count is called the cardinal number of the set. An infinite set is one which is not finite.

The following list of examples illustrates the distinction between finite and infinite sets:

1. If A is the set of days in the month of December, then set A is finite.
2. If set $B = \{1, 3, 5, 7, \cdots\}$, then set B is infinite.

3. If set $C = \{x | x$ is a grain of sand on the earth$\}$, then set C is finite. (The counting process would be difficult but would come to an end.)

4. If set $D = \{x | x$ is an animal on the earth$\}$, then set D is finite.

EQUALITY OF SETS

Sets A and B are said to be equal if and only if every element of A is an element of B and every element of B is an element of A. In this case we write

$$A = B$$

If

$$A = \{2, 3, 4, 5\}$$

and

$$B = \{5, 3, 4, 2\}$$

then

$$A = B$$

Notice that each of the elements 2, 3, 4, and 5 in A is in B and each of the elements in B is in A although the order of elements is different. The arrangement of elements does not change the set. Also, the set does not change if some elements are repeated.
If

$$A = \{3, 9, 9, 7\}$$

and

$$B = \{3, 3, 9, 7\}$$

then

$$A = B$$

In some cases the equality of sets is not obvious, as shown in the following:
If

$$A = \{x | x^2 - 5x = -6\}$$

and

$$B = \{2, 3\}$$

and

$$C = \{x | 1 < x < 4, \text{ x an integer}\}$$

then

$$A = B = C$$

NULL SET

We define the null set as a set which has no members or elements. This is a set which is void or empty. We denote this set by the symbol $\emptyset$ or $\{\ \}$. Notice that $\emptyset$ is not the same as 0; that is, 0 is not a set but $\emptyset$ is. Also, $\emptyset$ is not the same as $\{0\}$ because $\emptyset$ is the null set and $\{0\}$ is a set with the one element 0. The empty or null set may also be defined by a statement which prohibits any element from being a member of the set; that is, the set exists but has no members.

This is similar to regarding zero as a number; that is, the natural numbers are 1, 2, 3, $\cdots$ and are used for counting, and zero is not a counting number but it is used to indicate that there is nothing to count.

There is only one empty or null set because two sets are equal if they consist of the same elements, and since the empty sets have no members they are equal.

If

$$A = \{x | \text{ x is a 300-year-old man on earth}\}$$

then, as far as we know, A is the null set.

If

$$B = \{x | x^2 = 9, \text{ x is even}\}$$

then

$$B = \text{the null set}$$

or

$$B = \emptyset \text{ or } \{\ \}$$

PROBLEMS:

1. Use set notation to rewrite the following statements:
   a. x does not belong to C.
   b. k is a member of B.

2. Write the following, using set-builder notation.
   a. $B = \{2, 4, 6, 8, \cdots\}$.
   b. C is the set of men in the Navy

3. Indicate which sets are finite.
   a. The days of the week.
   b. $\{x|x \text{ is an odd integer}\}$.
   c. $\{3, 6, 9, \cdots\}$.

4. Which pairs of sets are equal?
   a. $\{1, 2, 3\}$ and $\{2, 1, 3, 2\}$
   b. $\{k, 1, x\}$ and $\{x, k, n, 1\}$

5. Which of the following describe the null set?
   a. $C = \{x|x + 6 = 6\}$.
   b. $B = \{x|x \text{ is a positive integer less than one}\}$.

ANSWERS:

1. a. $x \notin C$
   b. $k \in B$

2. a. $B = \{x|x \text{ is even}\}$
   b. $C = \{x|x \text{ is a man in the Navy}\}$

3. a. finite
   b. infinite
   c. infinite

4. a. equal
   b. unequal

5. a. not the null set
   b. the null set

SUBSETS

We say that set B is a subset of set A if and only if every element of set B is an element of set A.

If we have the situation where

$$A = \{1, 2, 3\}$$

and

$$B = \{1, 2\}$$

then B is a subset of A and we write

$$B \subset A$$

Notice that every element of B is an element of A; that is,

$$x \in B \rightarrow x \in A$$

where the symbol → means "implies" or if the first ($x \in B$) is true then the second ($x \in A$) is true. Also, it should be noted that the null set is a subset of every set.

If

$$D = \{x|x \text{ is an odd integer}\}$$

and

$$E = \{1, 3, 5, 7\}$$

then

$$E \subset D$$

PROPER SUBSET

If we have two sets such that

$$F = G$$

we may write

$$F \subset G$$

and

$$G \subset F$$

and we say F is a subset of G and G is a subset of F.

If we write

$$K = K$$

we say K is a subset of itself.

Since there is a distinction between these subsets and the subsets previously mentioned, we may call the previous subsets "proper subsets"; that is, B is a proper subset of A if B is a subset of A and at the same time B is not equal to A.

When we have

$$C = \{3, 4, 5\}$$

and

$$D = \{3, 4, 5\}$$

then C is a subset of D and we properly write

$$C \subseteq D$$

which indicates that C is also equal to D.

Although this distinction is made in some studies of sets and subsets, we will not make any distinction in the following discussions.

COMPARABILITY

If we have two sets where A is a subset of B or B is a subset of A, then we say sets A and B are comparable; that is, if

$$A \subset B$$

or

$$B \subset A$$

then A and B are comparable.

For two sets to be noncomparable we must have the relations

$$A \not\subset B$$

and

$$B \not\subset A$$

where the symbol $\not\subset$ means "is not a subset of."

If

$$C = \{3, 5, 6\}$$

and

$$D = \{5, 6\}$$

then C and D are comparable. This is because

$$D \subset C$$

If

$$E = \{7, 8, 9\}$$

and

$$F = \{8, 9, 10\}$$

then E and F are noncomparable. This is because there is an element in E not in F and there is an element in F not in E. This is written

$$7 \in E \quad \text{and} \quad 7 \notin F$$

and

$$10 \notin E \quad \text{and} \quad 10 \in F$$

UNIVERSAL SET

When we investigate sets whose elements are natural numbers, we say the natural numbers comprise the universal set. Generally, we say the universe is the set of natural numbers. We denote the universal set by the letter U.

If we are discussing sets of the letters of our alphabet, we call the alphabet the universe.

If we are talking about humans, then the universal set consists of all people on earth.

POWER SET

If we have a set A such that

$$A = \{1, 2, 4\}$$

and we list each subset of A; that is,

$$\{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{\ \}$$

we call this family of sets the power set of A.

The power set of any set S is the family of all subsets of S. This is denoted by

$$2^S$$

This designation is used because the number of subsets of any finite set of n elements is 2^n. If we have the set B such that

$$B = \{8, 9, 10\}$$

then, the power set of B; that is,

$$2^B$$

has

$$\begin{aligned} 2^n &= 2^3 \\ &= 8 \end{aligned}$$

subsets.

The power set is shown as follows:
If

$$B = \{8, 9, 10\}$$

then

$$2^B = \left\{ B, \{8,9\}, \{8,10\}, \{9,10\}, \{8\}, \{9\}, \{10\}, \emptyset \right\}$$

DISJOINT SETS

When we find two sets that have no elements in common, we say these sets are disjoint.

If

$$A = \{1, 2, 3\}$$

and

$$B = \{4, 5, 6\}$$

then A and B are disjoint.

If

$$C = \{a, b, c\}$$

and

$$D = \{c, d, e\}$$

then C and D are not disjoint because

$$c \in C \quad \text{and} \quad c \in D$$

VENN-EULER DIAGRAMS

The use of Venn-Euler diagrams, or simply Venn diagrams, is not an acceptable "proof" of the relationships among sets. Nevertheless we will use these diagrams for our intuitive approach to these relationships. We will use a circle to denote a set and a rectangle to indicate the universe. If we have two sets such that

$$A = \{1, 2, 3\}$$

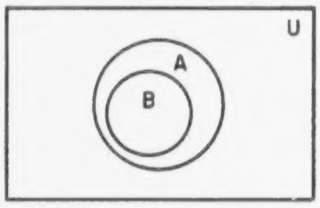
and

$$B = \{2, 3\}$$

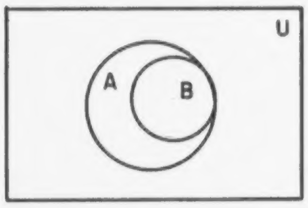
then we may show that

$$B \subset A$$

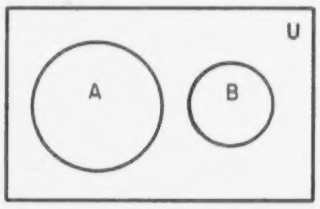
by drawing



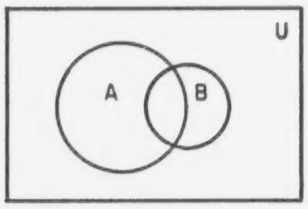
or



If we have two disjoint sets, we draw



If the sets are not disjoint, we draw



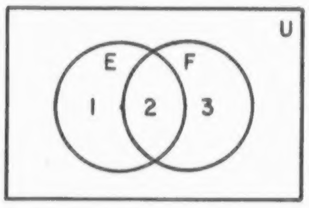
If

$$E = \{1, 2\}$$

and

$$F = \{2, 3\}$$

then we show their relationship by



indicating that

$1 \in E, \quad 1 \notin F$

$2 \in E, \quad 2 \in F$

$3 \notin E, \quad 3 \in F$

PROBLEMS:

1. If set $A = \{2, 3, 7, 9\}$, then how many subsets does A contain?

2. If set $B = \{x | x$ is an integer between 5 and $8\}$, then how many subsets does B contain? Write the power set.

3. Indicate whether the following pairs of sets are comparable or not.

   a. $A = \{5, 6, 9\}$ and $B = \{9, 10, 11\}$
   b. $C = \{x | x \text{ is even}\}$ and
      $D = \{x | x^2 + 5x = -6\}$
   c. $E = \{x | x \text{ is odd}\}$ and $\emptyset$

4. If

$A = \{1\}$
$B = \{1, 2\}$
$C = \{2, 3, 4\}$
$D = \{3, 4\}$

and

$E = \{1, 3, 4\}$

then indicate whether the following are true or false.

a. $D \subset C$
b. $A \neq E$
c. $A \not\subset D$
d. $A \subset C$
e. $D \not\subset E$
f. $B \subset E$

5. Make a Venn diagram of the following relationships.

a. $C \subset B$, $B \subset A$
b. $C \subset B$, $D \subset B$, $B \subset A$, and C and D are disjoint.
c. $A = \{1, 2, 3\}$, $B = \{2, 4\}$
d. $A = \{1, 2, 3, 4\}$, $B = \{4\}$,
   $C = \{3, 4, 5\}$

ANSWERS:

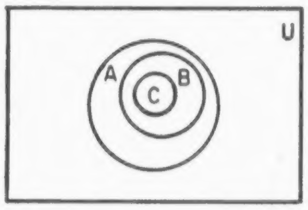
1. 2^4 or 16

2. 2^2 or 4; $\{B, \{6\}, \{7\}, \emptyset\}$

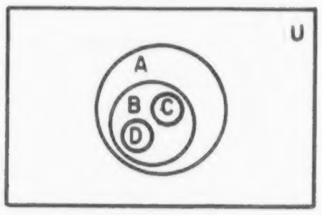
3. a. not comparable
   b. not comparable
   c. comparable

4. a. true
   b. true
   c. true
   d. false
   e. false
   f. false

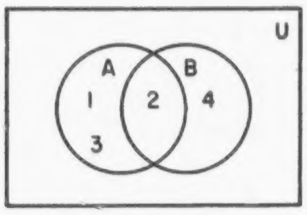
5. a.



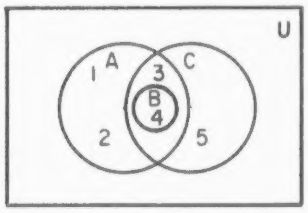
b.



c.



d.



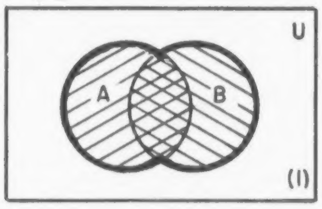
OPERATIONS

When operating with arithmetic, the process of adding a pair of numbers A and B produces a number A + B called the sum. Subtraction of B from A produces a number A - B called the difference and multiplication of A and B produces AB called the product. In this chapter we will discuss operations of sets which are somewhat similar to the arithmetic operations of addition, subtraction, and multiplication. The set operations are union, intersection, and difference. Complements will also be discussed.

The operations of sets will be discussed in relation to Venn diagrams which is a "show" rather than a "prove" type approach. The laws of sets will be discussed later in the chapter.

UNION

We say that the union of two sets A and B is the set of all elements which belong to A or B or to both A and B. We indicate the union of A and B by writing $A \cup B$. To show this union by a Venn diagram we draw diagram (1)



where the circles show the sets A and B and the rectangle indicates the universe U. We shaded set A with positive slope lines and shaded set B with negative slope lines. Any part of the universe which is shaded is "A union B" or $A \cup B$; that is,

$$A \cup B = \{x | x \in A \quad \text{or} \quad x \in B\}$$

An example using numerals to show the union of two sets A and B is as follows:
If

$$A = \{1, 2, 3, 4\}$$

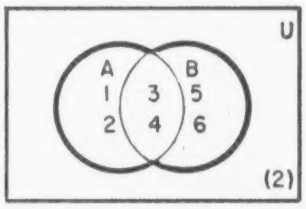
and

$$B = \{3, 4, 5, 6\}$$

then

$$A \cup B = \{1, 2, 3, 4, 5, 6\}$$

To show this union by Venn diagram, we draw diagram (2)



From the previous discussion it should be apparent that

$$A \cup B = B \cup A$$

In a later discussion we will relate the union of A and B to A + B.

PROBLEMS:

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8\}$, and $C = \{3, 5, 6\}$

Find

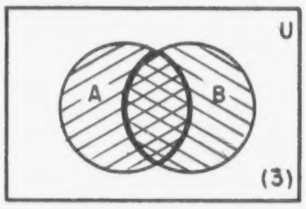
1. $A \cup B$
2. $A \cup C$
3. $B \cup C$
4. $A \cup A$
5. $(A \cup B) \cup C$

ANSWERS:

1. $\{1, 2, 3, 4, 5, 6, 8\}$
2. $\{1, 2, 3, 4, 5, 6\}$
3. $\{2, 3, 4, 5, 6, 8\}$
4. $\{1, 2, 3, 4, 5\}$
5. $\{1, 2, 3, 4, 5, 6, 8\}$

INTERSECTION

We say that the intersection of two sets A and B is the set of all elements which belong to A and B by writing $A \cap B$; that is, $A \cap B = \{x | x \in A, x \in B\}$. To show this intersection by a Venn diagram we draw (3)



(3)

A and B have been shaded as before and the intersection of A and B, that is, $A \cap B$ or the area shaded by cross-hatch.

An example using numerals to show the intersection of two sets A and B is as follows:

If

$$A = \{1, 2, 3, 4\}$$

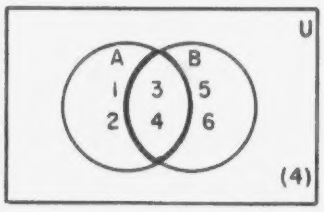
and

$$B = \{3, 4, 5, 6\}$$

then

$$A \cap B = \{3, 4\}$$

To show this intersection by Venn diagram we draw (4)



(4)

Again, it should be apparent that

$$A \cap B = B \cap A$$

In a later discussion we will relate the intersection of A and B to $A \cdot B$.

PROBLEMS:

Let $A = \{1, 3, 5\}$, $B = \{1, 2, 3, 4\}$, and $C = \{3, 4, 5, 6, 7\}$

Find

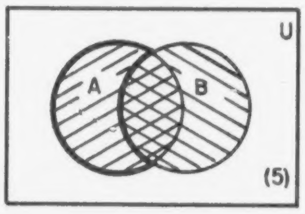
1. $A \cap B$
2. $A \cap C$
3. $B \cap C$
4. $A \cap A$
5. $(A \cap B) \cap C$

ANSWERS:

1. $\{1, 3\}$
2. $\{3, 5\}$
3. $\{3, 4\}$
4. $\{1, 3, 5\}$
5. $\{3\}$

DIFFERENCE

We say that the difference of two sets A and B is the set of elements which belong to A but do not belong to B. We indicate this by writing A - B; that is, $A - B = \{x | x \in A, x \notin B\}$. To show this difference by a Venn diagram we draw (5)



(5)

A minus B is the area which contains the positive slope shading only.

An example using numerals is as follows: If

$$A = \{1, 2, 3, 4\}$$

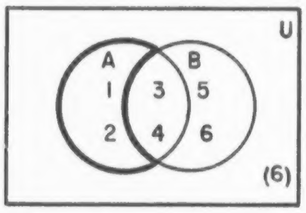
and

$$B = \{3, 4, 5, 6\}$$

then

$$A - B = \{1, 2\}$$

This is shown in Venn diagram form by drawing (6)



PROBLEMS:

Let $A = \{1, 3, 5\}$, $B = \{2, 4, 5\}$, and $C = \{3, 5\}$

Find

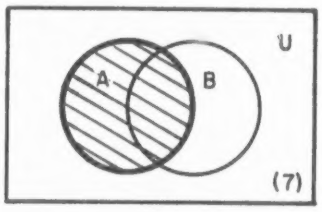
1. $A - B$
2. $A - C$
3. $B - C$
4. $B - A$
5. $A - A$

ANSWERS:

1. $\{1, 3\}$
2. $\{1\}$
3. $\{2, 4\}$
4. $\{2, 4\}$
5. $\{\ \}$ or $\emptyset$

COMPLEMENT

We say that the complement of a set A is the set of all elements within the universe which do not belong to A. This is comparable to the universe U minus the set A. We indicate the complement of A by writing $\overline{A}$; that is, $\overline{A} = \{x \mid x \in U, x \notin A\}$. Shown in Venn diagram form this is (7)



The area which is not shaded is the complement of A; that is, $\overline{A}$.

An example using numerals to show the complement of A is as follows:

If

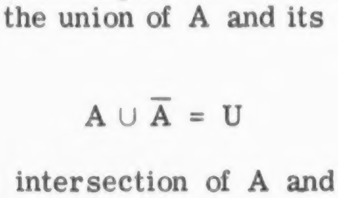
$$A = \{1, 2, 3, 4\} \quad \text{and} \quad B = \{3, 4, 5, 6\}$$

then

$$\overline{A} = \{5, 6, 7, 8, \cdots\}$$

where we assume the universe U to be the set of natural numbers 1, 2, 3, $\cdots$.

Shown in Venn diagram this is (8)



where the shaded area is the complement of A.

If we now use the previous information given, we find that the union of A and its complement $\overline{A}$ is

$$A \cup \overline{A} = U$$

Also, the intersection of A and its complement A is

$$A \cap \overline{A} = \emptyset$$

and the complement of the universal set U is the empty set $\emptyset$; that is,

$$\overline{U} = \emptyset$$

and

$$\overline{\emptyset} = U$$

PROBLEMS:

Let $A = \{2,3,4\}$, $B = \{2,4,6\}$, $C = \{4,5,6\}$, and the universe $U = \{1, 2, 3, \cdots, 10\}$.
Find

1. $\overline{A}$
2. $\overline{B}$
3. $\overline{(A \cup C)}$
4. $\overline{(A \cap B)}$
5. $\overline{(B - C)}$

ANSWERS:

1. $\{1, 5, 6, 7, 8, 9, 10\}$
2. $\{1, 3, 5, 7, 8, 9, 10\}$
3. $\{1, 7, 8, 9, 10\}$
4. $\{1, 3, 5, 6, 7, 8, 9, 10\}$
5. $\{1, 3, 4, 5, 6, 7, 8, 9, 10\}$

LAWS OF ALGEBRA OF SETS

We will discuss the relations that exist between sets and the operations of union, intersection, and complements. These operations satisfy various laws or identities which are called the algebra of sets.

We will use an intuitive approach to understanding these laws and in most cases will show the laws by use of Venn diagrams.

IDEMPOTENT LAWS

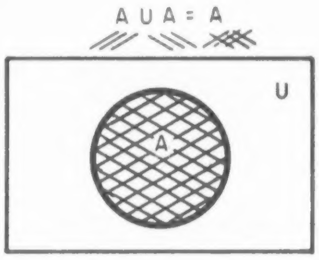
The relations $A \cup A = A$ and $A \cap A = A$ are the idempotent identities. Since the union of two sets A and B is $\{x | x \in A \text{ or } x \in B\}$, it follows that

$$A \cup A = A$$

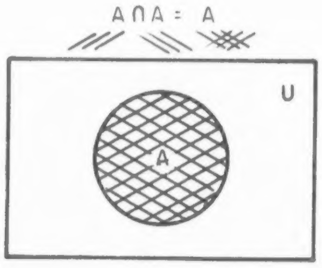
Also, since the intersection of two sets A and B is $\{x | x \in A,\ x \in B\}$, it follows that

$$A \cap A = A$$

Venn diagrams to show this are as follows: (We show each A and how it is shaded.)



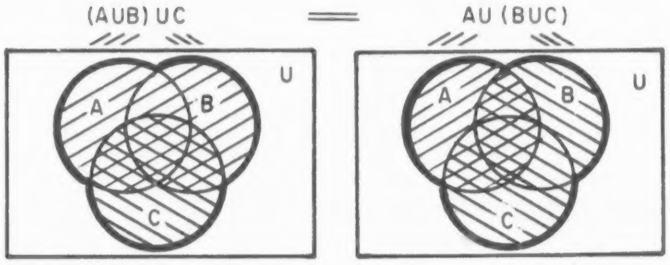
and



In both cases the cross-hatch is the solution area.

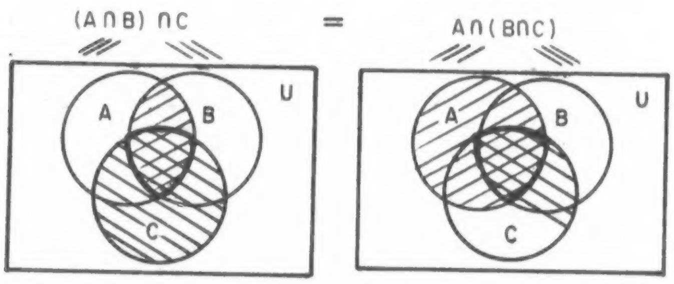
ASSOCIATIVE LAWS

The relations $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ are the associative identities. We show these as follows:



where the area with any shading is the same in each case.

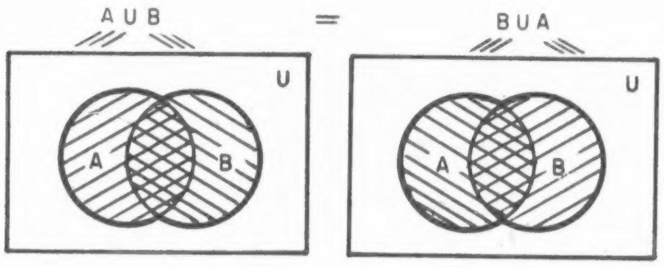
Also,


(A∩B) ∩C
=
A∩(B∩C)
A
B
C
U

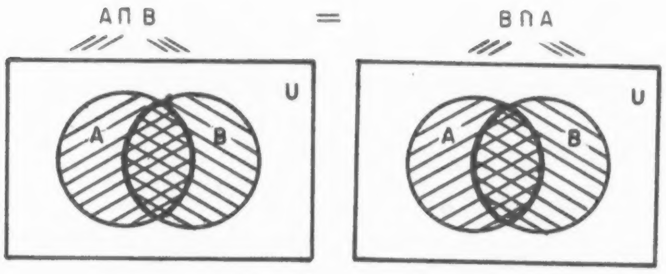
where the cross-hatched areas are the same.

COMMUTATIVE LAWS

The relations $A \cup B = B \cup A$ and $A \cap B = B \cap A$ are the commutative identities. They are shown as follows:

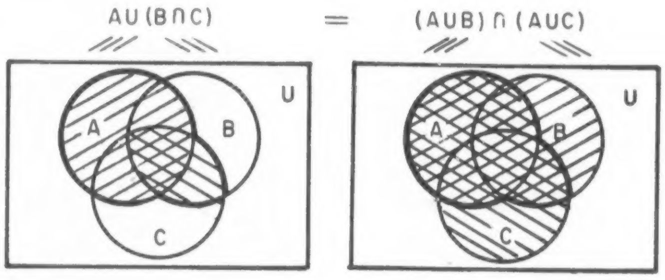


and

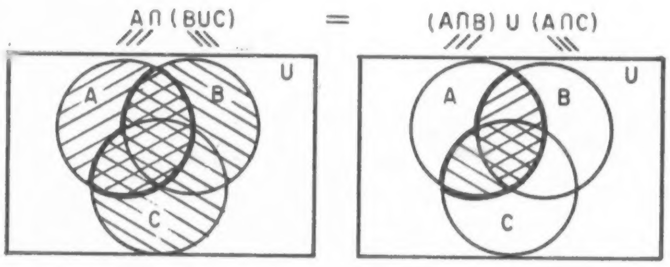


DISTRIBUTIVE LAWS

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ are the distributive identities. They indicate that $\cup$ distributes over $\cap$ and that $\cap$ distributes over $\cup$. They are shown as follows:

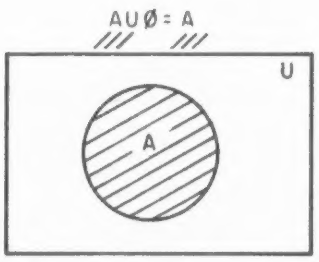

A∪(B∩C)
=
(A∪B) ∩ (A∪C)
A
B
C
U

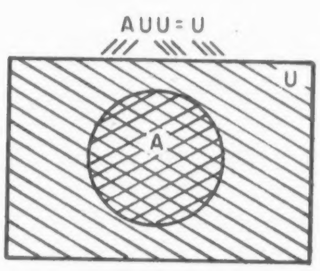
and


A∩ (B∪C)
=
(A∩B) ∪ (A∩C)
A
B
C
U

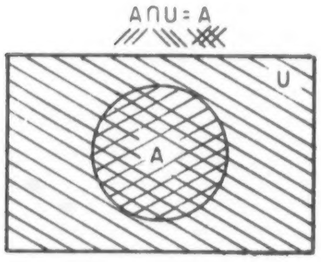
IDENTITY LAWS

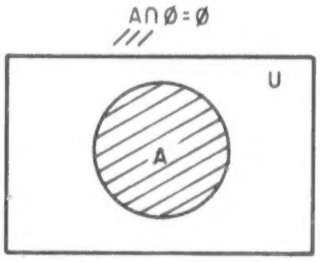
These laws state that $A \cup \emptyset = A$, $A \cup U = U$, $A \cap U = A$, and $A \cap \emptyset = \emptyset$. The Venn diagrams for these are shown as follows:





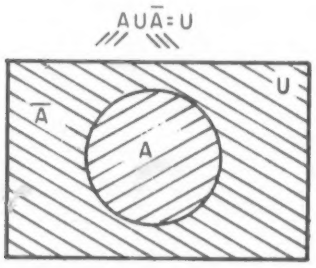
and

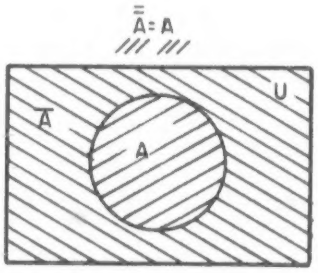


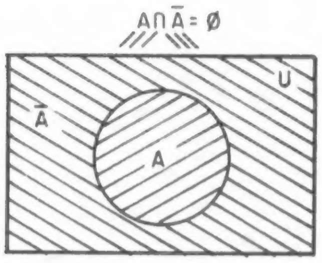


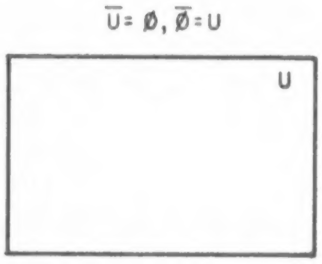
COMPLEMENT LAWS

These laws state that $A \cup \overline{A} = U$, $\overline{\overline{A}} = A$, $A \cap \overline{A} = \emptyset$, and $\overline{U} = \emptyset$, $\overline{\emptyset} = U$. These are shown as follows:



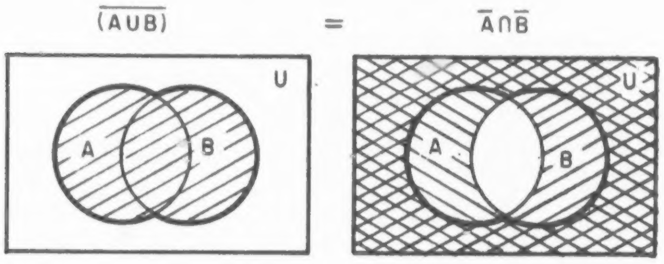




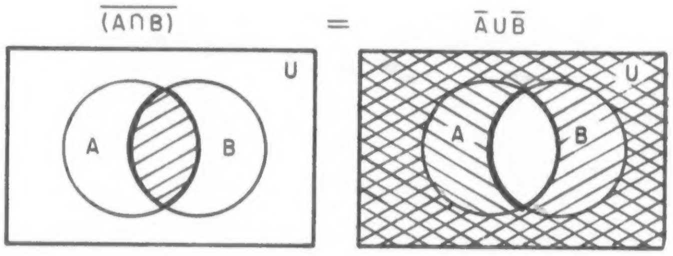


DE MORGAN'S LAWS

DeMorgan's laws are indicated by $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ and $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$. They are shown in Venn diagram form by the following:



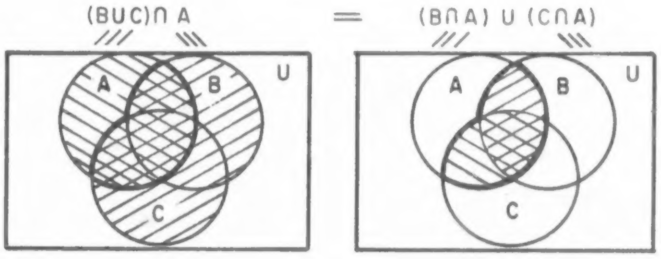
where $\overline{(A \cup B)}$ is the unshaded area and $\overline{A} \cap \overline{B}$ is the crosshatched area. Also,



where $\overline{(A \cap B)}$ is the unshaded area and $\overline{A} \cup \overline{B}$ is the shaded area.

PRINCIPLE OF DUALITY

This principle of the theory of sets states that if we interchange the operations of union and intersection and also interchange the universe and null set in any theorem then the new equation is a valid theorem. We may show this by the following:



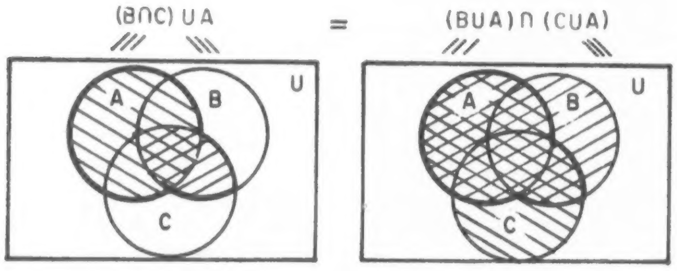
Now, the dual of

$$(B \cup C) \cap A = (B \cap A) \cup (C \cap A) \qquad (1)$$

is

$$(B \cap C) \cup A = (B \cup A) \cap (C \cup A) \qquad (2)$$

Since we have shown equation (1) to be true, then by the principle of duality equation (2) is true. We may show equation (2) to be true by writing



The following is a summary of the laws of the algebra of sets and the principle of duality:

Idempotent Laws

$A \cup A = A$ $\qquad$ $A \cap A = A$

Associative Laws

$(A \cup B) \cup C = A \cup (B \cup C)$ $\qquad$ $(A \cap B) \cap C = A \cap (B \cap C)$

Commutative Laws

$A \cup B = B \cup A$ $\qquad$ $A \cap B = B \cap A$

Distributive Laws

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\qquad$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Identity Laws

$A \cup \emptyset = A$ $\qquad$ $A \cap U = A$

$A \cup U = U$ $\qquad$ $A \cap \emptyset = \emptyset$

Complement Laws

$A \cup \bar{A} = U$ $\qquad$ $A \cap \bar{A} = \emptyset$

$\bar{\bar{A}} = A$ $\qquad$ $\bar{U} = \emptyset, \bar{\emptyset} = U$

DeMorgan's Laws

$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$ $\qquad$ $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

Principle of Duality

$(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$

$(B \cap C) \cup A = (B \cup A) \cap (C \cup A)$

The following examples illustrate the use of some of the laws of the algebra of sets to change an expression from one form to another. The particular law used in each step is indicated.

EXAMPLE: Prove that

$$C \cap (B \cup A) = (C \cap B) \cup (C \cap A).$$

SOLUTION: Write	Law used
$C \cap (B \cup A) = (C \cap B) \cup (C \cap A)$	distributive
$= (C \cap A) \cup (C \cap B)$	commutative
$= (C \cap B) \cup (C \cap A)$	commutative

EXAMPLE: Prove that

$$(A \cup B) \cap (B \cup C) = (A \cap C) \cup B.$$

SOLUTION: Write	Law used
$(A \cup B) \cap (B \cup C)$	
$= (B \cup A) \cap (B \cup C)$	commutative
$= B \cup (A \cap C)$	distributive
$= (A \cap C) \cup B$	commutative

EXAMPLE: Prove that

$$(A \cap B) \cup (A \cap \bar{B}) = A.$$

SOLUTION: Write	Law used
$(A \cap B) \cup (A \cap \bar{B})$	
$= A \cap (B \cup \bar{B})$	distributive
$= A \cap U$	complement substitution
$= A$	identity

EXAMPLE: Prove that $A \cup (\bar{A} \cap B) = A \cup B$.

SOLUTION: Write	Law used
$A \cup (\bar{A} \cap B)$	
$= (A \cup \bar{A}) \cap (A \cup B)$	distributive
$= U \cap (A \cup B)$	complement substitution
$= A \cup B$	identity

EXAMPLE: Prove that $A \cup (A \cap B) = A$.

SOLUTION: Write	Law used
$A \cup (A \cap B) = (A \cap U) \cup (A \cap B)$	identity substitution
$= A \cap (U \cup B)$	distributive
$= A \cap U$	identity substitution
$= A$	identity

EXAMPLE: Prove that $A \cap (A \cup B) = A$.

SOLUTION: Write	Law used
$A \cap (A \cup B) = (A \cup \emptyset) \cap (A \cup B)$	identity substitution
$= A \cup (\emptyset \cap B)$	distributive
$= A \cup \emptyset$	identity substitution
$= A$	identity

EXAMPLE: Prove that $(A \cup U) \cap (A \cap \emptyset) = \emptyset$.

SOLUTION: Write	Law used
$(A \cup U) \cap (A \cap \emptyset) = U \cap (A \cap \emptyset)$	identity substitution
$= U \cap \emptyset$	identity substitution
$= \emptyset$	identity

EXAMPLE: Prove that

$$\overline{A} \cup \overline{(B \cup C)} = \overline{(A \cap B)} \cap \overline{(A \cap C)}.$$

SOLUTION: Write	Law used
$\overline{A} \cup \overline{(B \cup C)} = \overline{A} \cup (\overline{B} \cap \overline{C})$	DeMorgan
$= (\overline{A} \cup \overline{B}) \cap (\overline{A} \cup \overline{C})$	distributive
$= \overline{(A \cap B)} \cap \overline{(A \cap C)}$	DeMorgan

EXAMPLE: Prove that $\overline{(\overline{A} \cup B)} = A \cap \overline{B}$.

SOLUTION: Write	Law used
$\overline{(\overline{A} \cup B)} = \overline{\overline{A}} \cap \overline{B}$	DeMorgan
$= A \cap \overline{B}$	complement

CHAPTER 7

BOOLEAN ALGEBRA

The father of Boolean algebra was George Boole, who was an English logician and mathematician. In the spring of 1847, he wrote a pamphlet on symbolic logic. Later he wrote a much larger text on which are founded the mathematical theories of logic. He did not regard logic as a branch of mathematics, but he did point out that a close analogy between symbols of algebra and those symbols which he devised to represent logical forms does exist.

Boolean algebra lay almost dormant until 1937 when Boole's algebra was used to write symbolic analyses of relay and switching circuits. Boolean algebra has now become an important subject to be learned in order to understand electronic computer circuits.

CLASSES AND ELEMENTS

We have previously determined that in our universe we can logically visualize two divisions; all things of interest in any discussion are in one division, and all other things not of interest are in the other division. These two divisions comprise a set or class called the UNIVERSAL CLASS. All objects contained in the universal class are called ELEMENTS. We also identify a set or class containing no elements; this class is called the NULL CLASS.

If we group some elements of the universal class together to form the combinations which are possible in a particular discussion, we call each of these combinations a class. In Boolean logic, these combinations called classes should not be confused with the null class or universal class. Actually, these classes are subclasses of the universal class. It should also be noted that the elements and classes in Boolean algebra are the sets and subsets previously discussed.

Each class is dependent upon its elements and the possible states (stable, nonstable, or both) that the elements can take.

Boolean algebra is that algebra which is based on Boolean logic and concerned with all elements having only two possible stable states and no unstable states.

To determine the number of classes or combinations of elements in Boolean algebra, we solve for the numerical value of 2^n where n equals the number of elements. If we have two elements (each element has two possible states) then we have 2^n or 2^2 possible classes. If we let the elements be A and B, then A may be true or false and B may be true or false. The classes which could be formed are as follows:

A true and B false
A true and B true
A false and B true
A false and B false

where we use the connective word "and." We could also form classes by use of the connective word "or" which would result in a different form of classes.

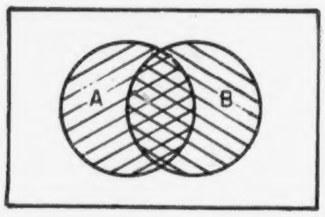
VENN DIAGRAMS

Since the Venn diagram is a topographical picture of logic, composed of the universal class divided into classes depending on the n number of elements, we show this logic as follows.

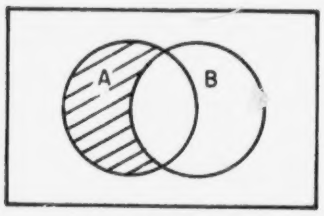
We may consider the universal class as containing submarines and atomic powered sound sources. Let A equal submarines and B equal atomic powered sound sources. Therefore, we have four classes which are:

(1) Submarines and not atomic
(2) Submarines and atomic
(3) Atomic and not submarines
(4) Not submarines and not atomic

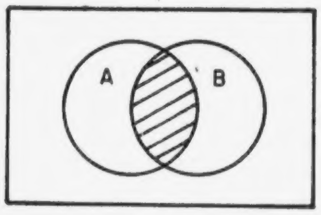
A diagram of these classes is



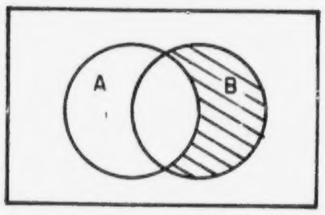
We may show these classes separately by



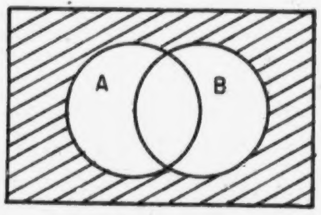
A and not B



A and B

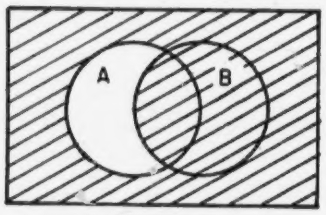


B and not A

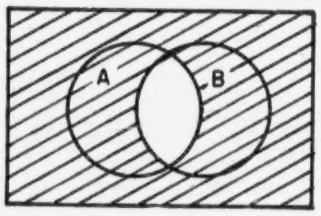


not A and not B

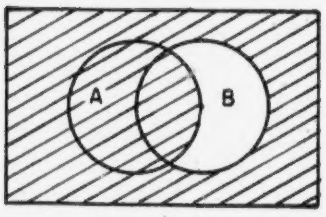
These four classes are called minterms because they represent the four minimum classes. The opposite of the minterms are called maxterms and are shown by



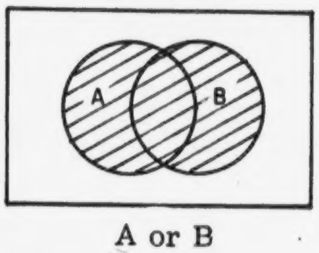
B or not A



not A or not B



A or not B



A or B

We will discuss minterms and maxterms in more detail later in the chapter.

BASIC EXPRESSIONS

It has been seen that the Venn diagram may be used to represent a picture of logic. The logic previously used was written in longhand, and used the words "and," "or," and "not." We used these words as a basis for combining elements to form classes in Boolean algebra logic descriptions. The symbols from sets and subsets are ∩ for "and," ∪ for "or," and — for "not." The relationships of symbols are given by the following:

Sets and subsets	Words	Boolean algebra
∩	and	·
∪	or	+
—	not	—

The following are examples of these relationships:

(1) $A \cdot B$ reads A and B
(2) $A + B$ reads A or B
(3) $\overline{A}$ reads not A

Relationships to the previously indicated classes, about submarines and atomic powered sound, are

(1) A and not B = $A \cdot \overline{B}$
(2) A and B = $A \cdot B$
(3) B and not A = $B \cdot \overline{A}$
(4) not A and not B = $\overline{A} \cdot \overline{B}$

also

(1) B or not A = $B + \overline{A}$
(2) not A or not B = $\overline{A} + \overline{B}$
(3) A or not B = $A + \overline{B}$
(4) A or B = $A + B$

Notice that

$A \cdot \overline{B}$
$A \cdot B$
$B \cdot \overline{A}$
$\overline{A} \cdot \overline{B}$

are called minterms. As related to algebra, there is a minimum number of terms in each; that is, one. Notice also that

$B + \overline{A}$
$\overline{A} + \overline{B}$
$A + \overline{B}$
$A + B$

are called maxterms. As related to algebra, there is a maximum number of terms in each. That is, two.

A further relationship may be made to sets and subsets as follows:

$$A \cdot \overline{B} = A \cap \overline{B}$$
$$A \cdot B = A \cap B$$
$$B \cdot \overline{A} = B \cap \overline{A}$$
$$\overline{A} \cdot \overline{B} = \overline{A} \cap \overline{B}$$

If we take any of these minterms, such as $A \cdot B$, and find its complement we have, according to DeMorgan's theorem

$$\begin{aligned} \overline{(A \cdot B)} &= \overline{(A \cap B)} \\ &= \overline{A} \cup \overline{B} \\ &= \overline{A} + \overline{B} \end{aligned}$$

which is a maxterm; therefore the complement of a minterm is a maxterm.

APPLICATIONS TO SWITCHING CIRCUITS

Since Boolean algebra is based upon elements having two possible stable states, it becomes very useful in representing switching circuits. The reason for this is that a switching circuit can be in only one of two possible states. That is, it is either open or it is closed. We may represent these two states as 0 and 1, respectively. Since the binary number system consists of only the symbols 0 and 1, we employ these symbols in Boolean algebra and call this "binary Boolean algebra."

THE "AND" OPERATION

Let us consider the Venn diagram in figure 7-1 (A). Its classes are labeled using the basic expressions of Boolean algebra. Note that there are two elements, or variables, A and B. The shaded area represents the class of elements that are $A \cdot B$ in Boolean notation and is expressed as:

$$f(A,B) = A \cdot B$$

The other three classes are also indicated in figure 7-1 (A). This expression is called an AND operation because it represents one of the four minterms previously discussed. Recall that AND indicates class intersection and both A and B must be considered simultaneously.

We can conclude then that a minterm of n variables is a logical product of these n variables with each variable present in either its noncomplemented or its complemented form, and is considered an AND operation.

For any Boolean function there is a corresponding truth table which shows, in tabular form, the true conditions of the function for each way in which conditions can be assigned its variables. In Boolean algebra, 0 and 1 are the symbols assigned to the variables of any function. Figure 7-1 (B) shows the AND operation function of two variables and its corresponding truth table.

This function can be seen to be true if one thinks of the logic involved: AB is equal to A and B which is the function f(A,B). Thus, if either A or B takes the condition of 0, or both take this condition, then the function f(A,B) equal AB is equal to 0. But if both A and B take the condition of 1 then the AND operation function has the condition of 1.

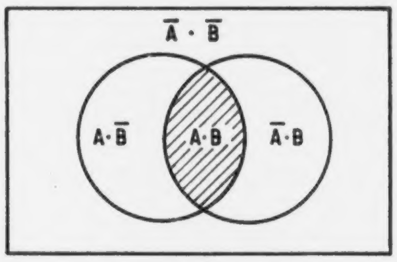
Figure 7-1 (C) shows a switching circuit for the function f(A,B) equal AB in that there will be an output only if both A and B are closed. An output in this case equals 1. If either switch is open, 0 condition, then there will be no output or 0.

In any digital computer equipment, there will be many circuits like the one shown in figure 7-1 (C). In order to analyze circuit operation, it is necessary to refer frequently to these circuits without looking at their switch arrangements. This is done by logic diagram mechanization as shown in figure 7-1 (D). This indicates that there are two inputs, A and B, into an AND operation circuit producing the function in Boolean algebra form of AB. These diagrams simplify equipment circuit diagrams by indicating operations without drawing all the circuit details.

It should be understood that while the previous discussion concerning the AND operation dealt with only two variables that any number of variables will fit the discussion. For example, in figure 7-2 three variables are shown along with their Venn diagram, truth table, switching circuit, and logic diagram mechanization.

THE "OR" OPERATION

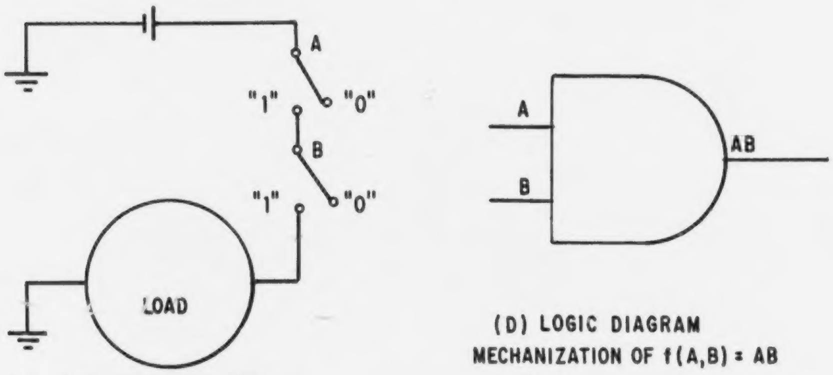
We will now consider the Venn diagram in figure 7-3 (A). Note that there are two elements, or variables, A and B. The shaded area represents the class of elements that are $A + B$


$\bar{A}\cdot\bar{B}$
$A\cdot\bar{B}$
$A\cdot B$
$\bar{A}\cdot B$

(A) VENN DIAGRAM

A	B	f(A,B) = AB
0	0	0
0	1	0
1	0	0
1	1	1

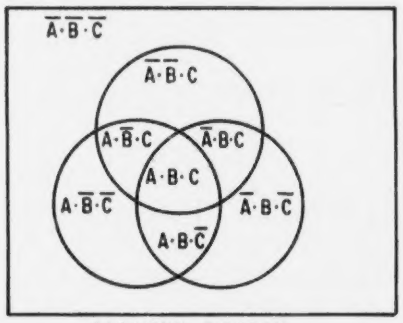
(B) TRUTH TABLE


A
"1"
"0"
B
"1"
"0"
LOAD
A
B
AB

(C) AND SWITCHING CIRCUIT

(D) LOGIC DIAGRAM MECHANIZATION OF f(A,B) = AB

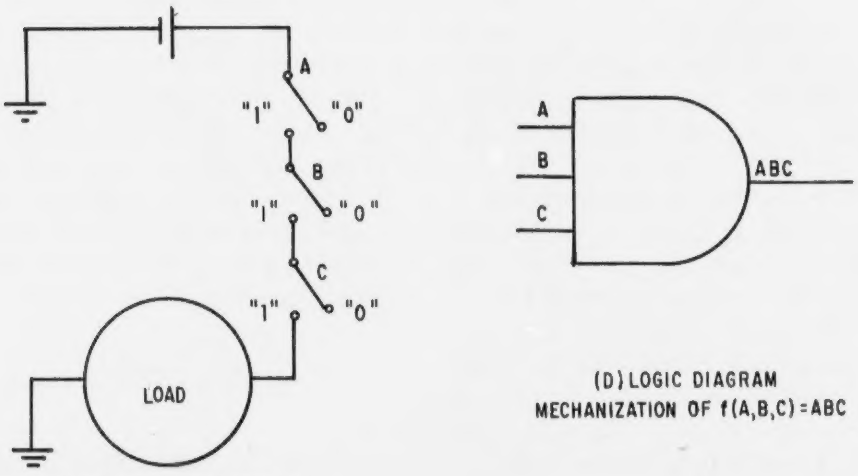
Figure 7-1.—The AND operation.


$\bar{A}\cdot\bar{B}\cdot\bar{C}$
$\bar{A}\cdot\bar{B}\cdot C$
$A\cdot\bar{B}\cdot C$
$\bar{A}\cdot B\cdot C$
$A\cdot B\cdot C$
$A\cdot\bar{B}\cdot\bar{C}$
$\bar{A}\cdot B\cdot\bar{C}$
$A\cdot B\cdot\bar{C}$

(A) VENN DIAGRAM

A	B	C	f(A,B,C) = ABC
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

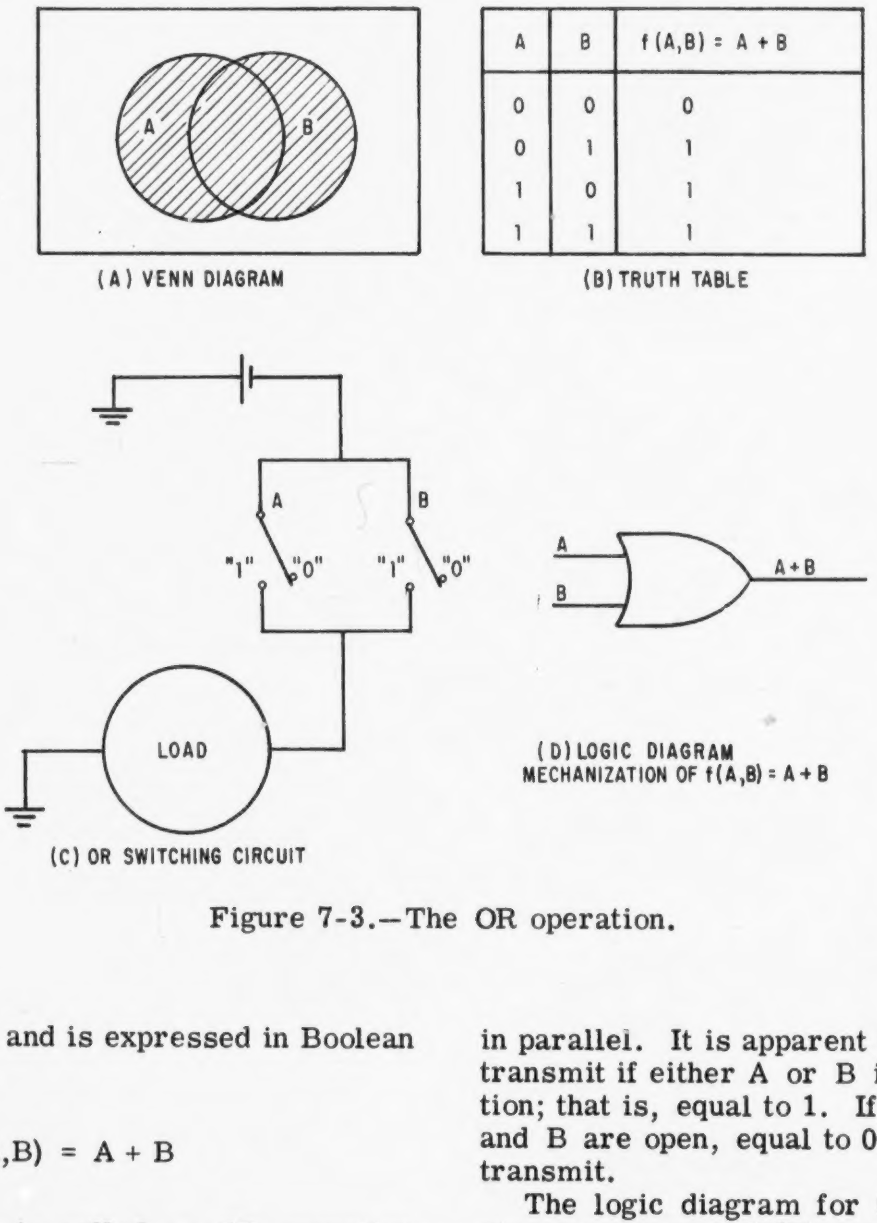
(B) TRUTH TABLE


A
"1"
"0"
B
"1"
"0"
C
"1"
"0"
LOAD
A
B
C
ABC

(C) AND SWITCHING CIRCUIT

(D) LOGIC DIAGRAM MECHANIZATION OF f(A,B,C) = ABC

Figure 7-2.—The AND operation (three variables).



(A) VENN DIAGRAM

A	B	f(A,B) = A + B
0	0	0
0	1	1
1	0	1
1	1	1

(B) TRUTH TABLE

(C) OR SWITCHING CIRCUIT

(D) LOGIC DIAGRAM MECHANIZATION OF f(A,B) = A + B

Figure 7-3.—The OR operation.

in Boolean notation and is expressed in Boolean algebra as:

$$f(A,B) = A + B$$

This expression is called an OR operation for it represents one of the four maxterms previously discussed. Recall that OR indicates class union and either A or B or both must be considered.

We can conclude then that a maxterm of n variables is a logical sum of these n variables where each variable is present in either its noncomplemented or its complemented form.

In figure 7-3 (B) the truth table of an OR operation is shown. This truth table can be seen to be true if one thinks of A + B being equal to A or B which is the function f(A,B). Thus if A or B takes the value 1, then f(A,B) must equal 1. If not, then the function equals zero.

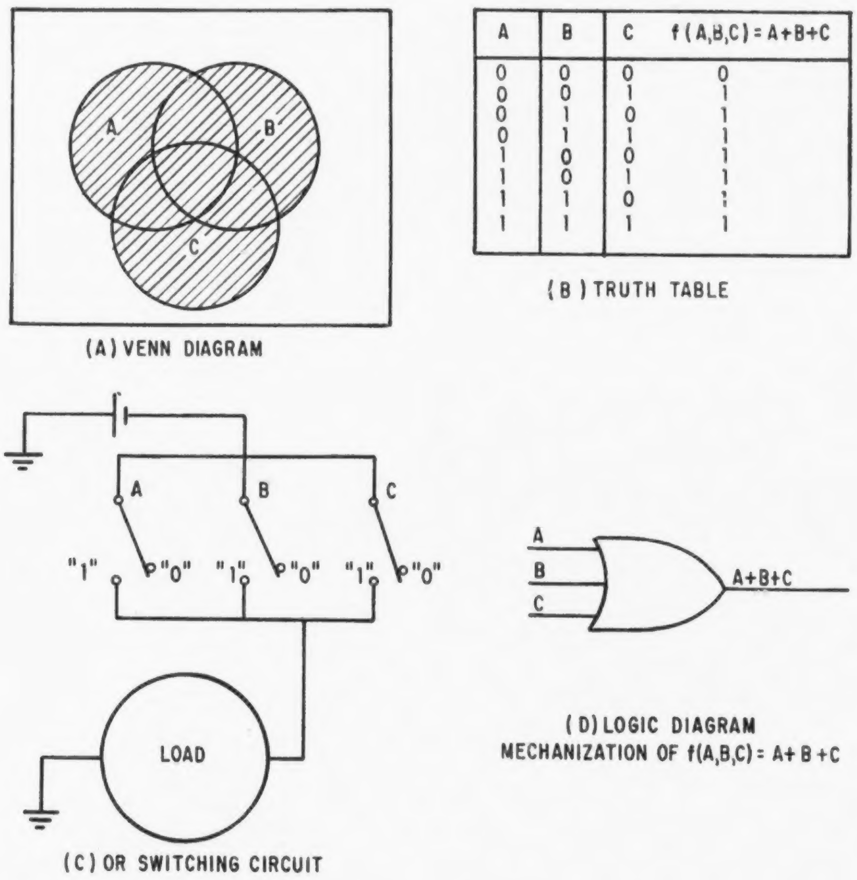
Figure 7-3 (C) shows a switching circuit for the OR operation which is two or more switches in parallel. It is apparent that the circuit will transmit if either A or B is in a closed position; that is, equal to 1. If, and only if, both A and B are open, equal to 0, the circuit will not transmit.

The logic diagram for the OR operation is given in figure 7-3 (D). This means that there are two inputs, A and B, into an OR operation circuit producing the function in Boolean form of A + B. Note the difference in the diagram from that of figure 7-2 (D).

As in the discussion of the AND operation the OR operation may also be used with more than two inputs. Figure 7-4 shows the OR operation with three inputs.

THE "NOT" OPERATION

The shaded area in figure 7-5 (A) represents the complement of A which in Boolean algebra is $\overline{A}$ and read as "NOT A." The expression f(A) equals $\overline{A}$ is called a NOT operation. The



A	B	C	f(A,B,C) = A+B+C
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

(A) VENN DIAGRAM

(B) TRUTH TABLE

(C) OR SWITCHING CIRCUIT

(D) LOGIC DIAGRAM MECHANIZATION OF f(A,B,C) = A+B+C

Figure 7-4.—The OR operation (three variables).

truth table for the NOT operation is explained by the NOT switching circuit. The requirement of a NOT circuit is that a signal injected at the input produce the complement of this signal at the output. Thus, in figure 7-5 (C) it can be seen that when switch A is closed, that is, equal to 1, the relay opens the circuit to the load. When switch A is open, that is, equal to 0, the relay completes a closed circuit to the load. The logic diagram for the NOT operation is given in figure 7-5 (D). This means that A is the input to a NOT operation circuit and gives an output of $\overline{A}$. The NOT operation may be applied to any operation circuit such as AND or OR. This is discussed in the following section.

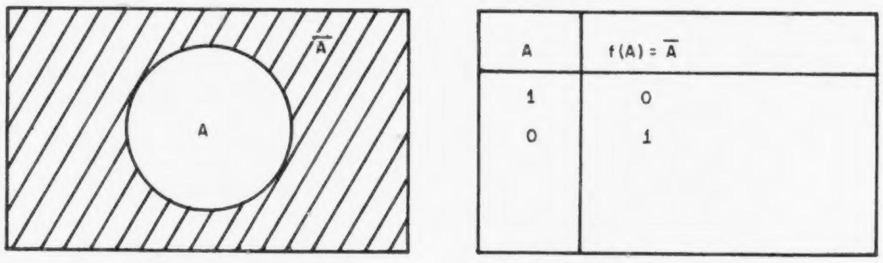
THE "NOR" OPERATION

The shaded area in figure 7-6 (A) represents the quantity, A OR B, negated. If reference is made to the preceding chapter it will be found that this figure is identical to the minterm expression $\overline{A}\overline{B}$; that is, A OR B negated is $\overline{A \text{ OR } B}$ and by application of DeMorgan's theorem is equal to $\overline{A}\overline{B}$.

The truth table for the NOR operation is shown in figure 7-6 (B). The table shows that if either A or B is equal to 1, then f(A,B) is equal to 0. Furthermore, if A and B equal 0, then f(A,B) equals 1.

The NOR operation is a combination of the OR operation and the NOT operation. The NOR switching circuit in figure 7-6 (C) is the OR circuit placed in series with the NOT circuit. If either switch A, switch B, or both are in the closed position, equal to 1, then there is no transmission to the load. If both switches A and B are open, equal to 0, then current is transmitted to the load.

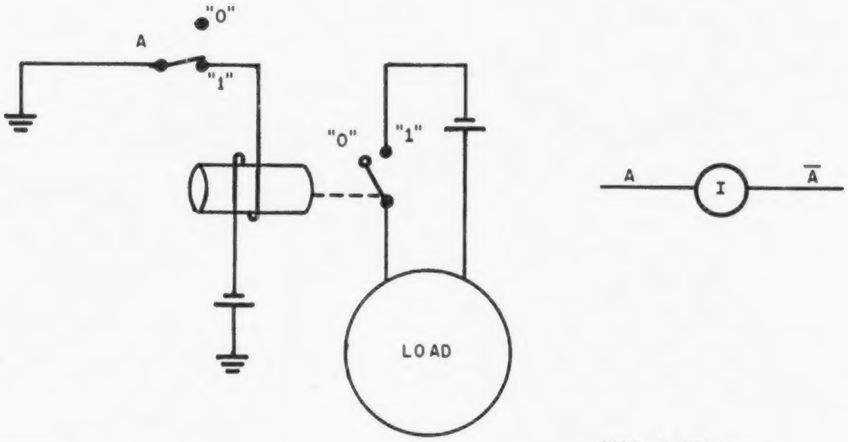
The logic diagram mechanization of f(A,B) equal $\overline{A+B}$ (NOR operation) is shown in figure 7-6 (D). It uses both the OR logic diagrams and the NOT logic diagrams. The NOR logic diagram mechanization shows there are two



A	$f(A) = \overline{A}$
1	0
0	1

(A) Venn Diagram

(B) Truth Table



(C) NOT Switching Circuit

(D) Logic Diagram Mechanization of $f(A)=\overline{A}$

Figure 7-5.--The NOT operation.

inputs, A and B, into an OR circuit producing the function in Boolean form of $A+B$. This function is the input to the NOT (inverter) which gives the output, in Boolean form, of $\overline{A+B}$. Note that the whole quantity of $A+B$ is complemented and not the separate variables.

THE "NAND" OPERATION

The shaded area in figure 7-7 (A) represents the quantity A AND B negated (NOT), and is a maxterm expression. Notice that $\overline{AB}$ is equal to the maxterm expression $\overline{A}+\overline{B}$.

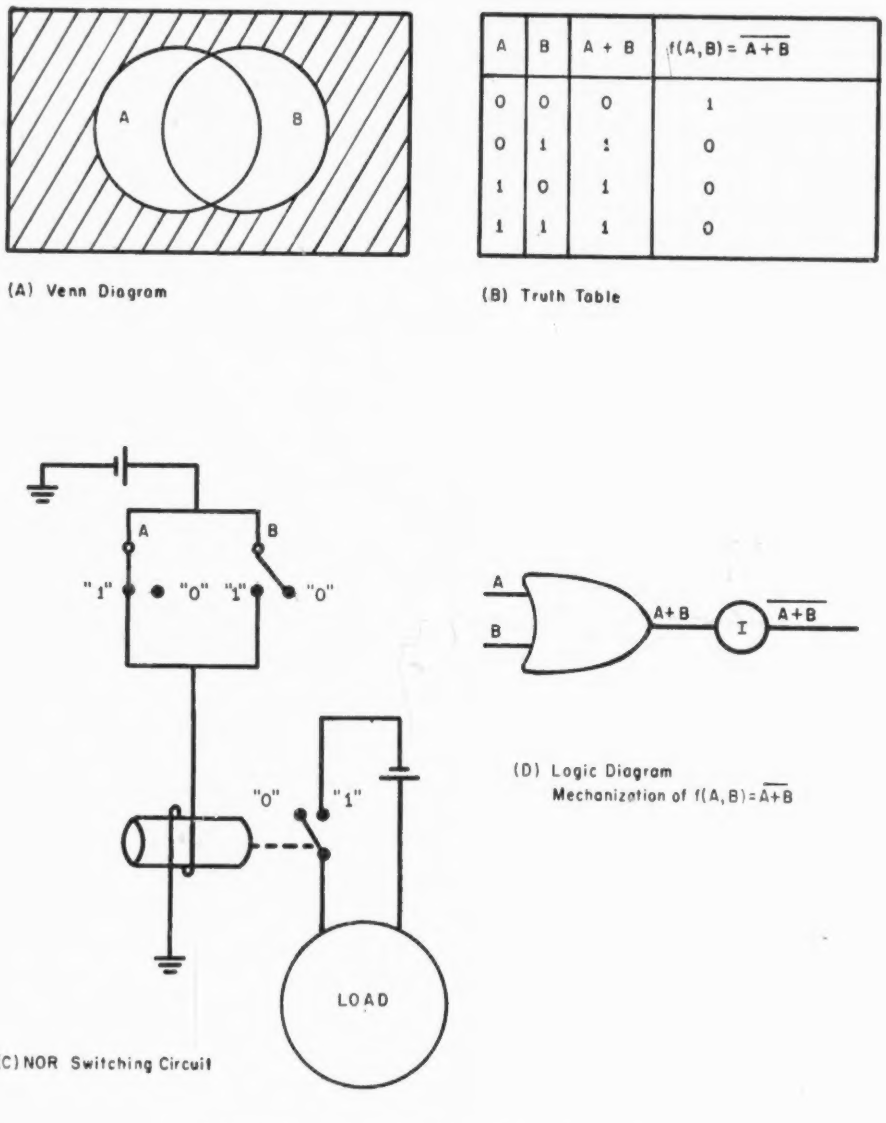
The truth table is shown for the NAND operation in figure 7-7 (B). When A and B equal 1, then f(A,B) is equal to 0. In all other cases, the function is equal to 1.

The NAND operation is a combination of the AND operation and the NOT operation. The NAND switching circuit in figure 7-7 (C) is the AND circuit put in series with the NOT circuit. If either switch A or B is open, equal to 0, then current is transmitted to the load. If both switch A and B are closed, equal to 1, then there is no transmission to the load.

The logic diagram mechanization of f(A,B) equal $\overline{AB}$ (NAND operation) is shown in figure 7-7 (D). The AND operation logic diagram and the NOT logic diagram mechanization shows that there are two inputs, A and B, into the AND circuit producing the function in Boolean form of AB. This function is the input to the NOT circuit which gives the output, in Boolean form, of $\overline{AB}$. Note that the entire quantity AB is complemented and not the separate variables.

It should be noted that in the previously discussed logic diagrams that each input signal represents the operation of a switch, circuit, or other component part.

Generally, a Boolean expression that has been inverted is said to be NOTTED. While we have previously used the inverter symbol



A	B	A + B	$f(A,B)=\overline{A+B}$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

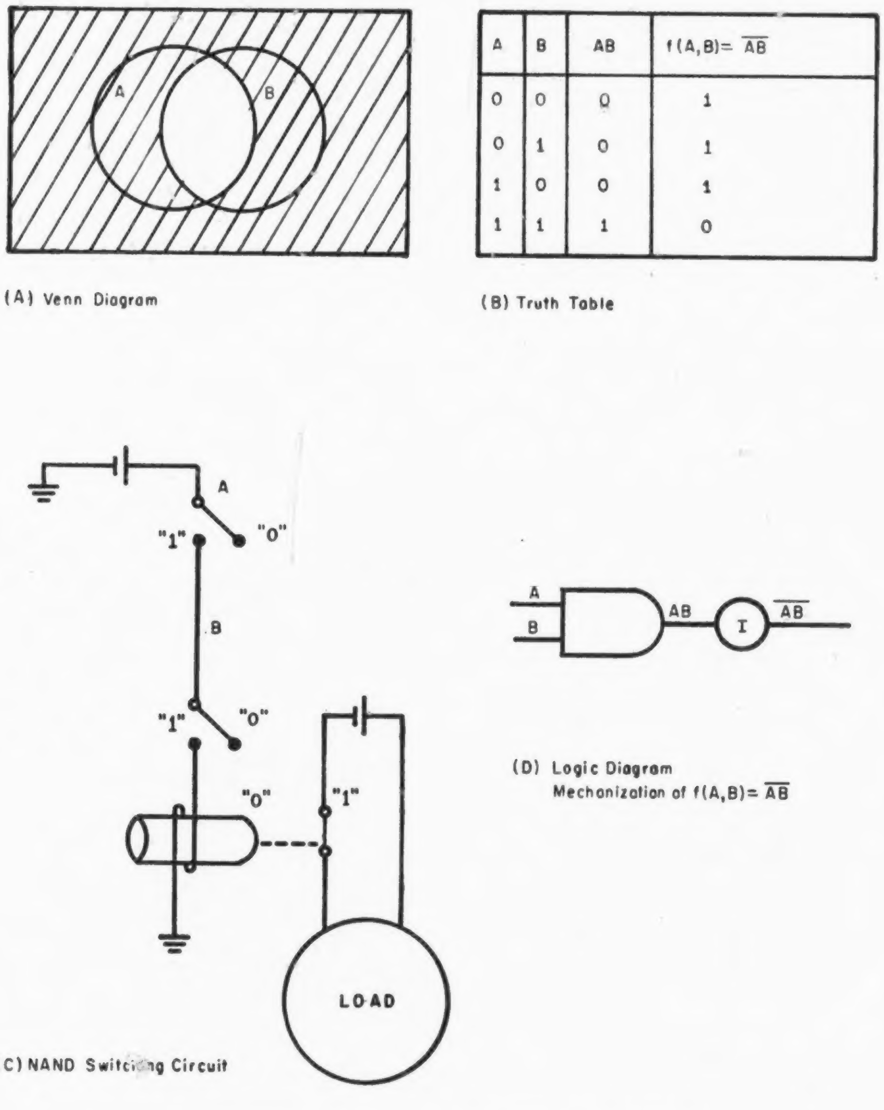
Figure 7-6.—The NOR operation.

separate from the AND or OR logic diagram it is common practice to show the NAND or NOR logic diagrams as indicated in figure 7-8, in accordance with American Standard for Graphic Symbols for Logic Diagrams (ASA Y32.14-1962).

The output of a NAND or a NOR gate is a NOTTED expression. The vinculum is used to indicate that such an expression has been NOTTED. Therefore, the output of a NAND gate having inputs A,B will appear as $\overline{AB}$ and the output of a NOR gate having inputs A,B will appear as $\overline{A+B}$. If any of the inputs to a logic gate are themselves NOTTED a vinculum will appear over the letter representing an input. Examples are shown in figure 7-8.

OUTPUT USED AS INPUT

The output from one gate may be an input to another gate. If so, that input will contain two or more letters. Figure 7-9 (A) shows an OR gate feeding into an OR gate. There are four possible combinations of inputs and logic symbols. These are shown in figure 7-9 (B,C,D,E). Notice that signs of grouping occur in all outputs except the AND input to the OR gate. The AB, in this case, is naturally grouped because the letters are written together and are separated from C by the OR sign. Figure 7-10 shows several different cases along with the proper output expressions.


A
B
"1"
"0"
"1"
"0"
"0"
"1"
LOAD
A
B
AB
I
$\overline{AB}$

(A) Venn Diagram

A	B	AB	$f(A,B)=\overline{AB}$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

(B) Truth Table

(C) NAND Switching Circuit

(D) Logic Diagram Mechanization of $f(A,B)=\overline{AB}$

Figure 7-7.—The NAND operation.

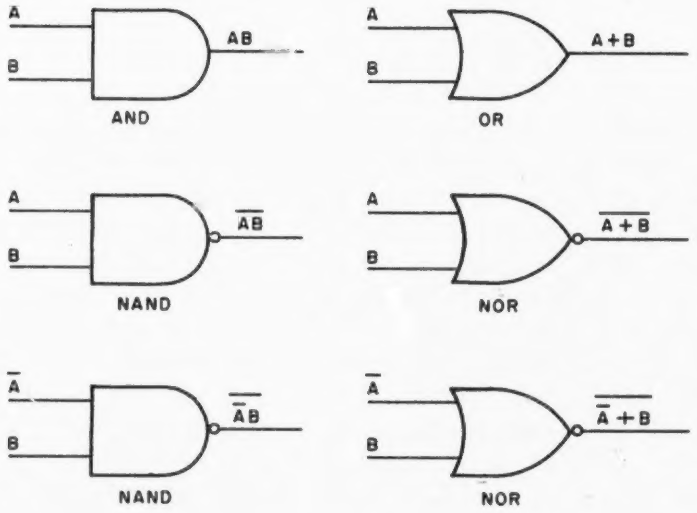

A
B
AB
AND
A
B
A+B
OR
A
B
$\overline{AB}$
NAND
A
B
$\overline{A+B}$
NOR
$\overline{A}$
B
$\overline{\overline{A}B}$
NAND
$\overline{A}$
B
$\overline{\overline{A}+B}$
NOR

Figure 7-8.—American Standard Logic Symbols.

Although the vinculum is not used in place of parentheses or brackets, it is also a grouping sign. Consider the NOR symbol of figure 7-11 (A). The AB and C are the inputs to the OR circuit and form AB + C. The AB + C is then inverted to form $\overline{AB+C}$. The vinculum groups whatever portion or portions of the output expression that has been inverted. Figure 7-11 (B,C,D) gives examples of this type output.

To determine the output of a logic diagram, find the output of each logic symbol in the diagram. You should begin with the inputs at the left and move right, using the output of each logic symbol as an input to the following symbol, as illustrated in figure 7-12.

When determining the output of a logic diagram, one should be careful of the two most

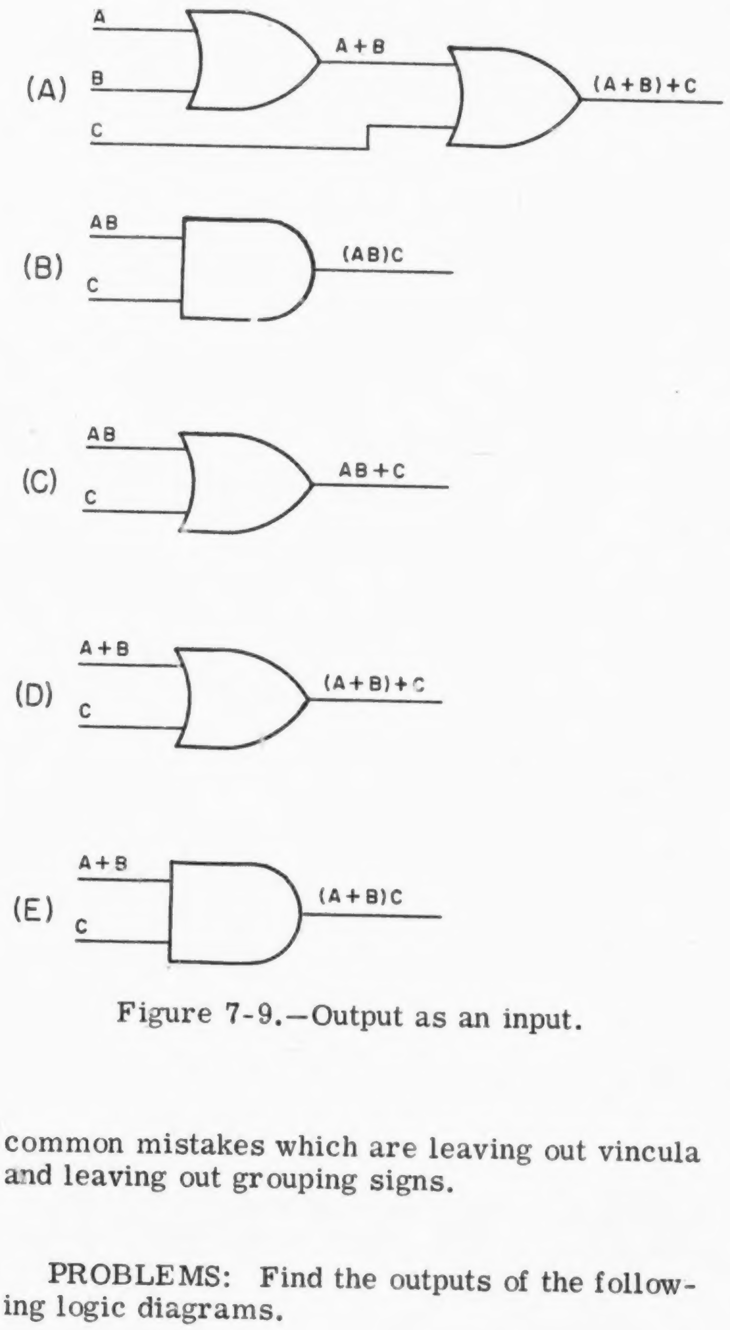
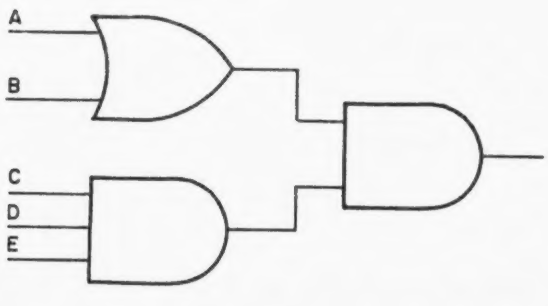

(A)
A
B
C
A+B
(A+B)+C
(B)
AB
C
(AB)C
(C)
AB
C
AB+C
(D)
A+B
C
(A+B)+C
(E)
A+B
C
(A+B)C

Figure 7-9.—Output as an input.

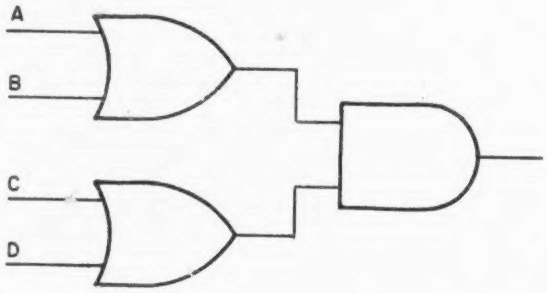
common mistakes which are leaving out vincula and leaving out grouping signs.

PROBLEMS: Find the outputs of the following logic diagrams.

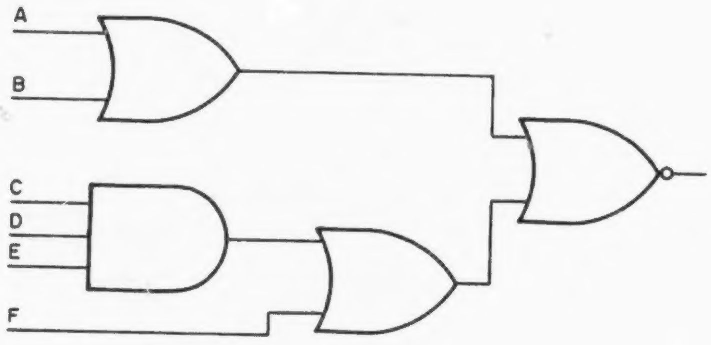
1.



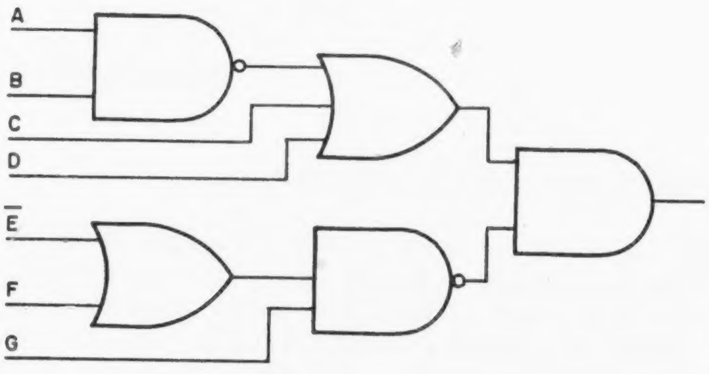
2.



3.



4.



ANSWERS:

1. $(A+B)(CDE)$
2. $(A+B)(C+D)$
3. $\overline{(A+B)+(CDE+F)}$
4. $(\overline{AB}+C+D)[G(\overline{E}+F)]$

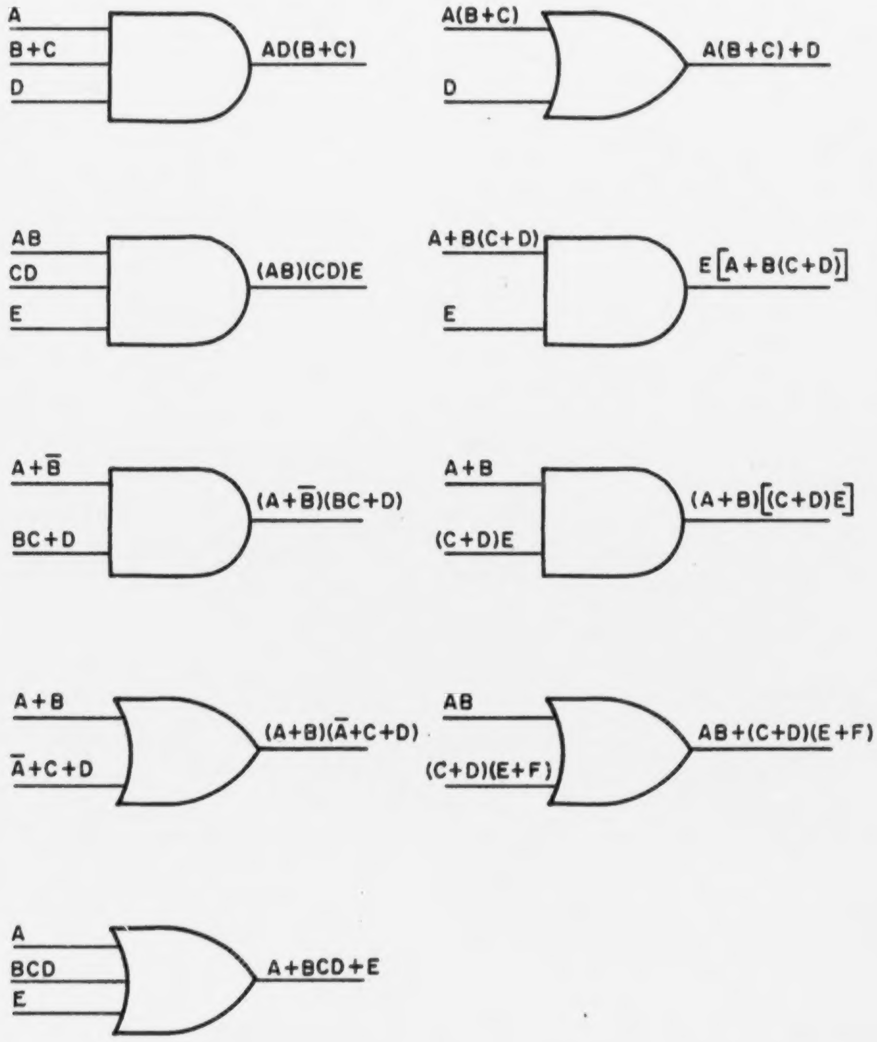
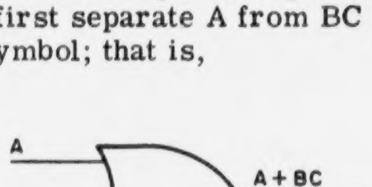

A
B+C
D
AD(B+C)
A(B+C)
D
A(B+C)+D
AB
CD
E
(AB)(CD)E
A+B(C+D)
E
E[A+B(C+D)]
$A+\overline{B}$
BC+D
$(A+\overline{B})(BC+D)$
A+B
(C+D)E
(A+B)[(C+D)E]
A+B
$\overline{A}+C+D$
$(A+B)(\overline{A}+C+D)$
AB
(C+D)(E+F)
AB+(C+D)(E+F)
A
BCD
E
A+BCD+E

Figure 7-10.—Examples of grouping.

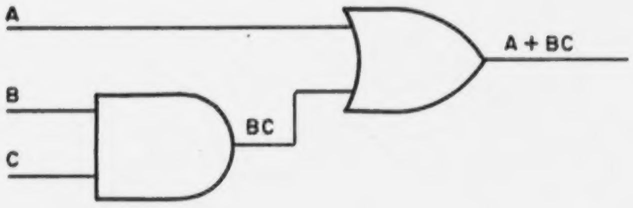
DEDUCING INPUTS FROM OUTPUTS

In order to draw a logic diagram from an output expression you should start with the output and work toward the input. Separate, in steps, the output expression until you have all single-letter inputs. If letters are grouped, first separate the group from other groups or letters, then separate the letters within groups.

To diagram the input that produces A + BC, you would first separate A from BC by using an OR logic symbol; that is,



You now draw an AND logic symbol to separate B from C, and extend all lines to a common column on the left. This is shown by the following diagram.



One common mistake in drawing the simplest possible diagram from an output expression is

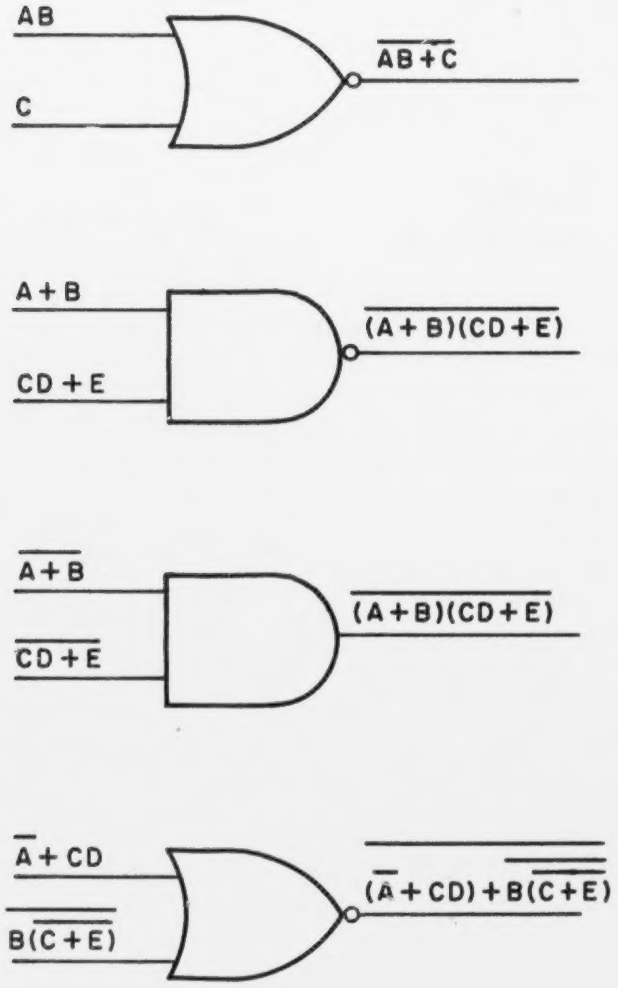
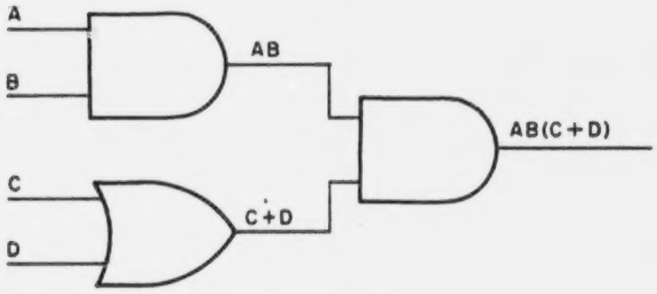
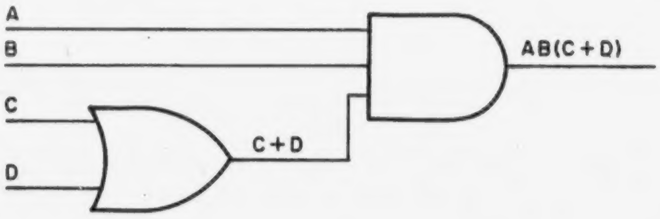

AB
C
$\overline{\overline{AB}+C}$
A + B
CD + E
$\overline{(A+B)(CD+E)}$
$\overline{A+B}$
$\overline{CD+E}$
$\overline{(A+B)(CD+E)}$
$\overline{A}$ + CD
$\overline{B(\overline{C+E})}$
$\overline{(\overline{A}+CD)+\overline{B(\overline{C+E})}}$

Figure 7-11.—Vinculum as grouping sign.

when the expression is similar to AB(C + D). The mistake is made by drawing

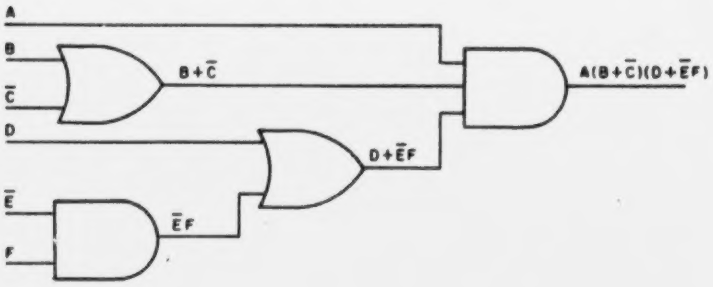

A
B
AB
AB(C + D)
C
D
C + D

If the foregoing were your results, you would have failed to notice that A, B, and (C + D) were all ANDed together. You should have drawn


A
B
AB(C + D)
C
C + D
D

which would have saved the use of one gate. A gate is considered one circuit such as OR, AND, NOR, etc.

To diagram the expression A(B + $\overline{C}$)(D + $\overline{E}$F) write

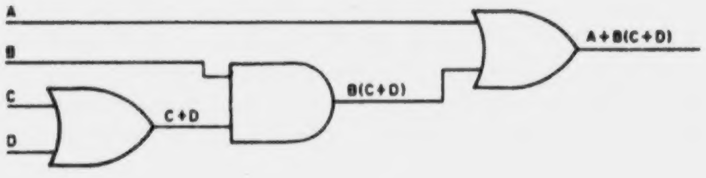

A
B
$\overline{C}$
B + $\overline{C}$
A(B + $\overline{C}$)(D + $\overline{E}$F)
D
D + $\overline{E}$F
$\overline{E}$
$\overline{E}$F
F

PROBLEMS: Draw the logic diagrams for the following expressions.

1. A + B(C + D)
2. (A + B + C)D + E
3. (A + B + C)DE
4. ABC(D + E)

ANSWERS:

1.


A
B
A + B(C + D)
B(C + D)
C
C + D
D

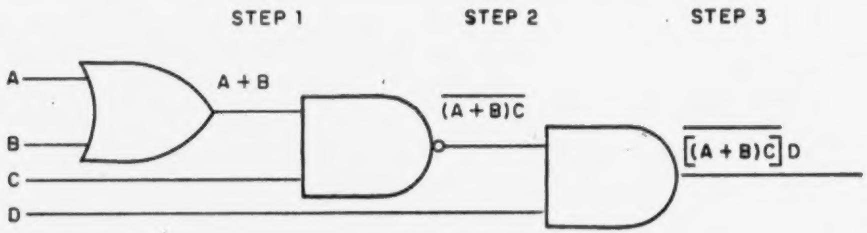
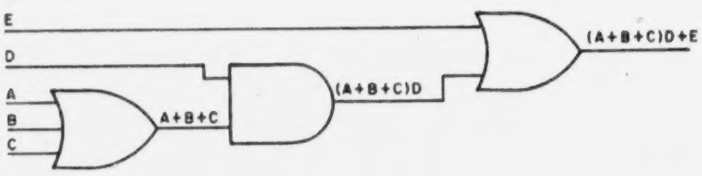
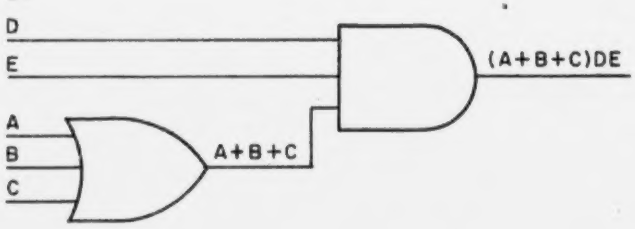

STEP 1
STEP 2
STEP 3
A
B
C
D
A + B
$\overline{(A+B)C}$
$\overline{[(A+B)C]}D$

Figure 7-12.—Steps for determining output.

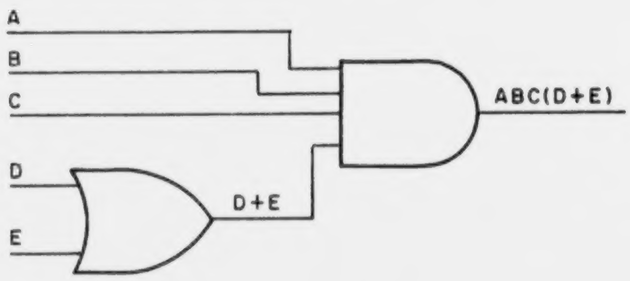
2.


E
D
A
B
C
A+B+C
(A+B+C)D
(A+B+C)D+E

3.


D
E
A
B
C
A+B+C
(A+B+C)DE

4.


A
B
C
D
E
D+E
ABC(D+E)

POSTULATES AND THEOREMS

In this section we will discuss the basic laws of Boolean algebra which enables one to simplify many Boolean expressions. By applying the basic laws, the digital systems designer can be sure a circuit is in the simplest possible algebraic form. The actual application of the basic laws will be discussed in the following chapter.

The laws of Boolean algebra may also be used for trouble-shooting defective components or for locating errors in computer programs. It should be understood that not all of the laws are similar to the laws of ordinary algebra.

LAW OF IDENTITY

This law is shown as

$$A = A$$

$$\bar{A} = \bar{A}$$

and indicates that any letter, number, or expression is equal to itself. The law of identity is shown in figure 7-13.

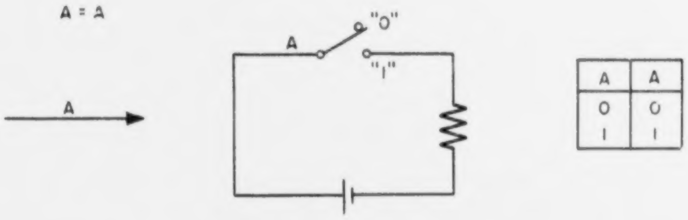

A = A
A
A
"0"
"1"
A
A
0
0
1
1

Figure 7-13.—Law of Identity.

COMMUTATIVE LAW

The commutative law is:

$$AB = BA$$

and

$$A + B = B + A$$

which is shown in figure 7-14. This indicates that when inputs to a logic symbol are ANDed

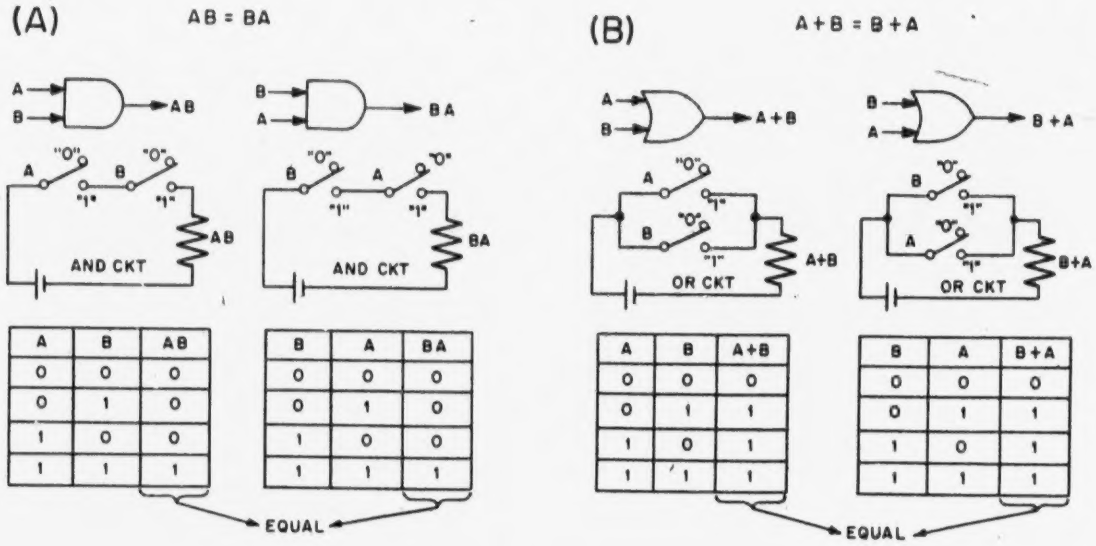


Figure 7-14.—Commutative Law.

or ORed, the order in which they are written does not affect the binary value of the output; that is,

$$R(S+T) = (S+T)R$$

and

$$A(BC+D+E) = (E+BC+D)A$$

ASSOCIATIVE LAW

The associative law is:

$$A(BC) = (AB)C$$

and

$$A + (B+C) = (A+B) + C$$

which is shown in figure 7-15. This indicates that when inputs to a logic symbol are ANDed or ORed, the order in which they are grouped does not affect the binary value of the output; that is,

$$ABC + D(EF) = (AB)C + DEF$$

and

$$C + (D+E) + (F+G) = C + D + E + F + G$$

IDEMPOTENT LAW

As seen in figure 7-16, if A is ANDed with A or if A is ORed with A, the output will equal A; that is,

$$AA = A$$

$$A + A = A$$

and

$$(RS)(RS) = RS$$

LAW OF DOUBLE NEGATION

This law is:

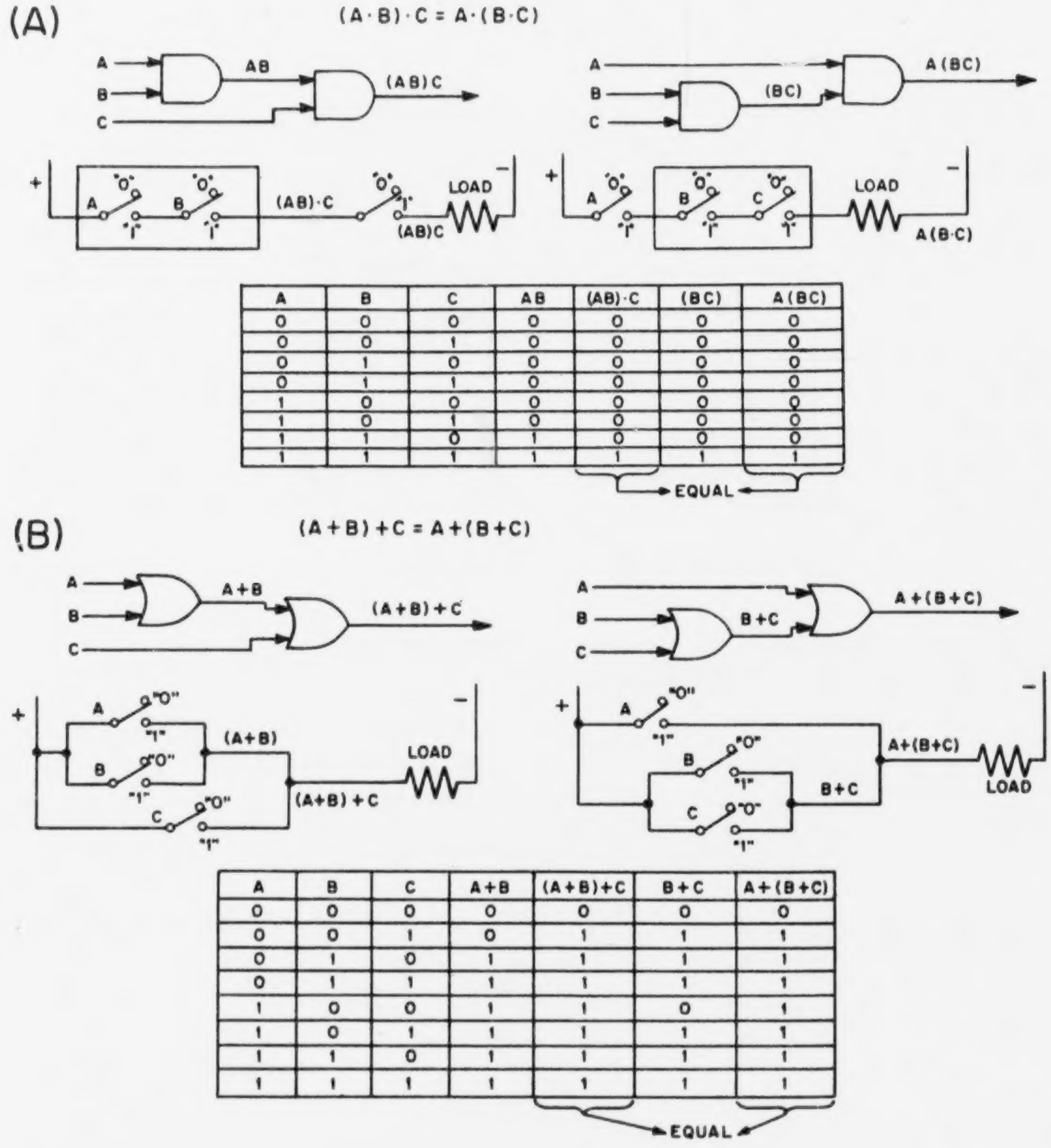
$$\overline{\overline{A}} = A$$

which indicates that when two bars of equal length cover the same letter or expression, both may be removed. This is shown in figure 7-17. Examples are

$$\overline{\overline{AB}} = AB$$

and

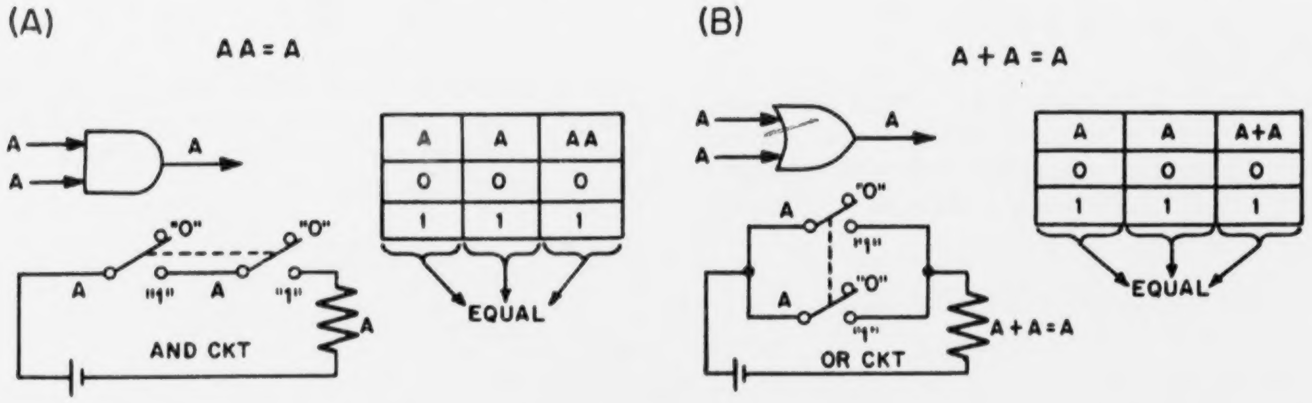
$$\overline{\overline{\overline{AB}}} + \overline{\overline{X}} = \overline{AB} + X$$


(A)
(A·B)·C = A·(B·C)
A
B
C
AB
(AB)C
A(BC)
(BC)
+
−
"0"
"1"
(AB)·C
LOAD
A(B·C)
EQUAL
(B)
(A + B) + C = A + (B + C)
A+B
(A+B) + C
B+C
A+(B+C)
(A+B)
LOAD
EQUAL

A	B	C	AB	(AB)·C	(BC)	A(BC)
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	0	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

A	B	C	A+B	(A+B)+C	B+C	A+(B+C)
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	1	1	0	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

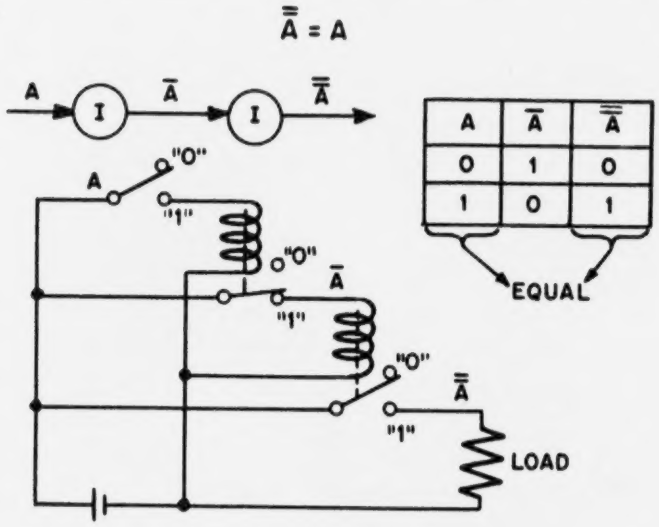
Figure 7-15.—Associative Law.


(A)
AA = A
A
"0"
"1"
AND CKT
EQUAL
(B)
A + A = A
A+A=A
OR CKT
EQUAL

A	A	AA
0	0	0
1	1	1

A	A	A+A
0	0	0
1	1	1

Figure 7-16.—Idempotent Law.


$\overline{\overline{A}} = A$
A
I
$\overline{A}$
I
$\overline{\overline{A}}$
"O"
"1"
"O"
"1"
"O"
"1"
$\overline{A}$
$\overline{\overline{A}}$
LOAD
EQUAL

A	$\overline{A}$	$\overline{\overline{A}}$
0	1	0
1	0	1

Figure 7-17.—Law of Double Negation.

COMPLEMENTARY LAW

This law is stated as:

$$A\overline{A} = 0$$

and

$$A + \overline{A} = 1$$

which indicates that when any letter or expression is ANDed with its complement, the output is 0. Also, when any letter or expression is ORed with its complement, the output is 1. This is shown in figure 7-18. Examples are:

$$CD\overline{CD} = 0$$

and

$$\overline{ABC} + ABC = 1$$

LAW OF INTERSECTION

As shown in figure 7-19, if one input to an AND circuit has a value of 1 the output will take the value of the other input. That is, if the two inputs to an AND circuit are 1 and A, then when A is 1 the output will be 1 and when A is 0 the output will be 0. If the inputs are 0 and A, then the output will always be 0.

The law of intersection is given by the following:

$$A \cdot 1 = A$$

and

$$A \cdot 0 = 0$$

Examples are:

$$AB \cdot 1 = AB$$

and

$$CD \cdot 0 = 0$$

LAW OF UNION

As shown in figure 7-20, if one input to an OR circuit has a binary value of 1, the output will be 1. If the inputs are 0 and A, the output will be the same as the value of A.

The law of union is given by the following:

$$A + 1 = 1$$

and

$$A + 0 = A$$

Examples of this law are as follows:

$$1 + ABC = 1$$

and

$$E + 0(AB) = E$$

LAW OF DUALIZATION
(DeMorgan's Theorem)

To split a vinculum that extends over more than one letter, and to join separate vincula into one vinculum requires the use of the law of dualization. This law is commonly referred to as DeMorgan's theorem. This law is shown in figure 7-21.

DeMorgan's theorem may be written as follows:

$$\overline{AB} = \overline{A} + \overline{B}$$

and

$$\overline{A + B} = \overline{A}\,\overline{B}$$

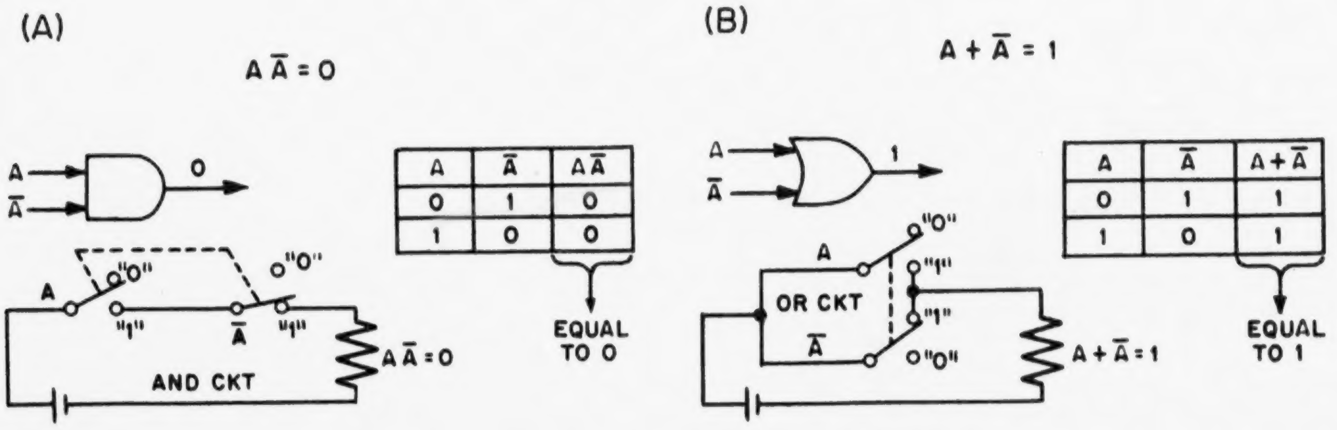

(A)
A Ā = 0
A
Ā
0
A | Ā | AĀ
0 | 1 | 0
1 | 0 | 0
EQUAL TO 0
A
"0"
"1"
Ā
"0"
"1"
AND CKT
A Ā = 0
(B)
A + Ā = 1
A
Ā
1
A | Ā | A+Ā
0 | 1 | 1
1 | 0 | 1
EQUAL TO 1
A
"0"
"1"
OR CKT
Ā
"1"
"0"
A + Ā = 1

Figure 7-18.—Complementary Law.

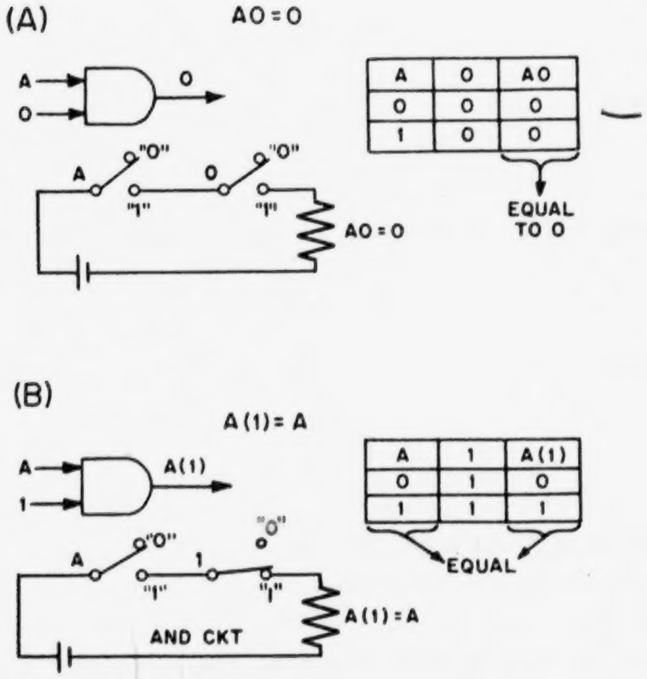

(A)
AO = O
A
O
O
A | O | AO
O | O | O
1 | O | O
EQUAL TO O
A
"O"
"1"
O
"O"
"1"
AO = O
(B)
A (1) = A
A
1
A(1)
A | 1 | A(1)
O | 1 | O
1 | 1 | 1
EQUAL
A
"O"
"1"
1
"O"
"1"
AND CKT
A (1) = A

Figure 7-19.—Law of Intersection.

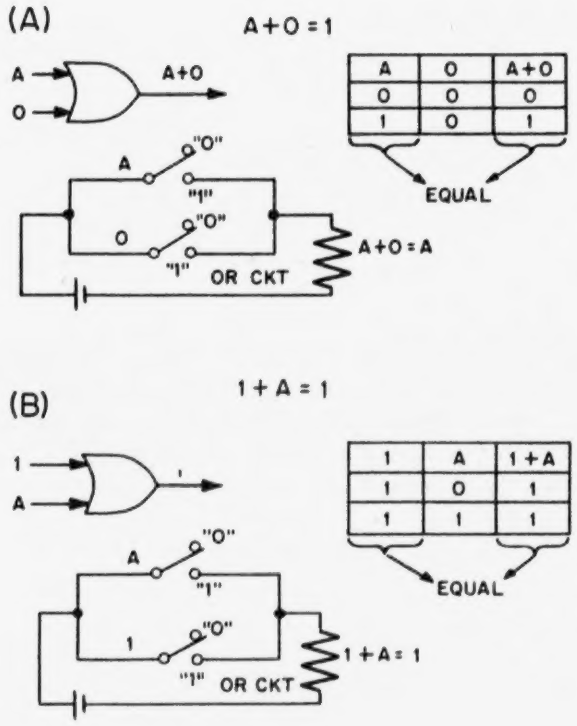

(A)
A+O = 1
A
O
A+O
A | O | A+O
O | O | O
1 | O | 1
EQUAL
A
"O"
"1"
O
"O"
"1"
OR CKT
A+O = A
(B)
1 + A = 1
1
A
1
1 | A | 1 + A
1 | O | 1
1 | 1 | 1
EQUAL
A
"O"
"1"
1
"O"
"1"
OR CKT
1 + A = 1

Figure 7-20.—Law of Union.

Whenever you split or join a vinculum, change the sign of operation. That is, AND to OR, or OR to AND.

In applying this theorem it should be remembered that when a vinculum covers part of an expression, the signs under the vinculum change and the signs outside the vinculum do not change; that is,

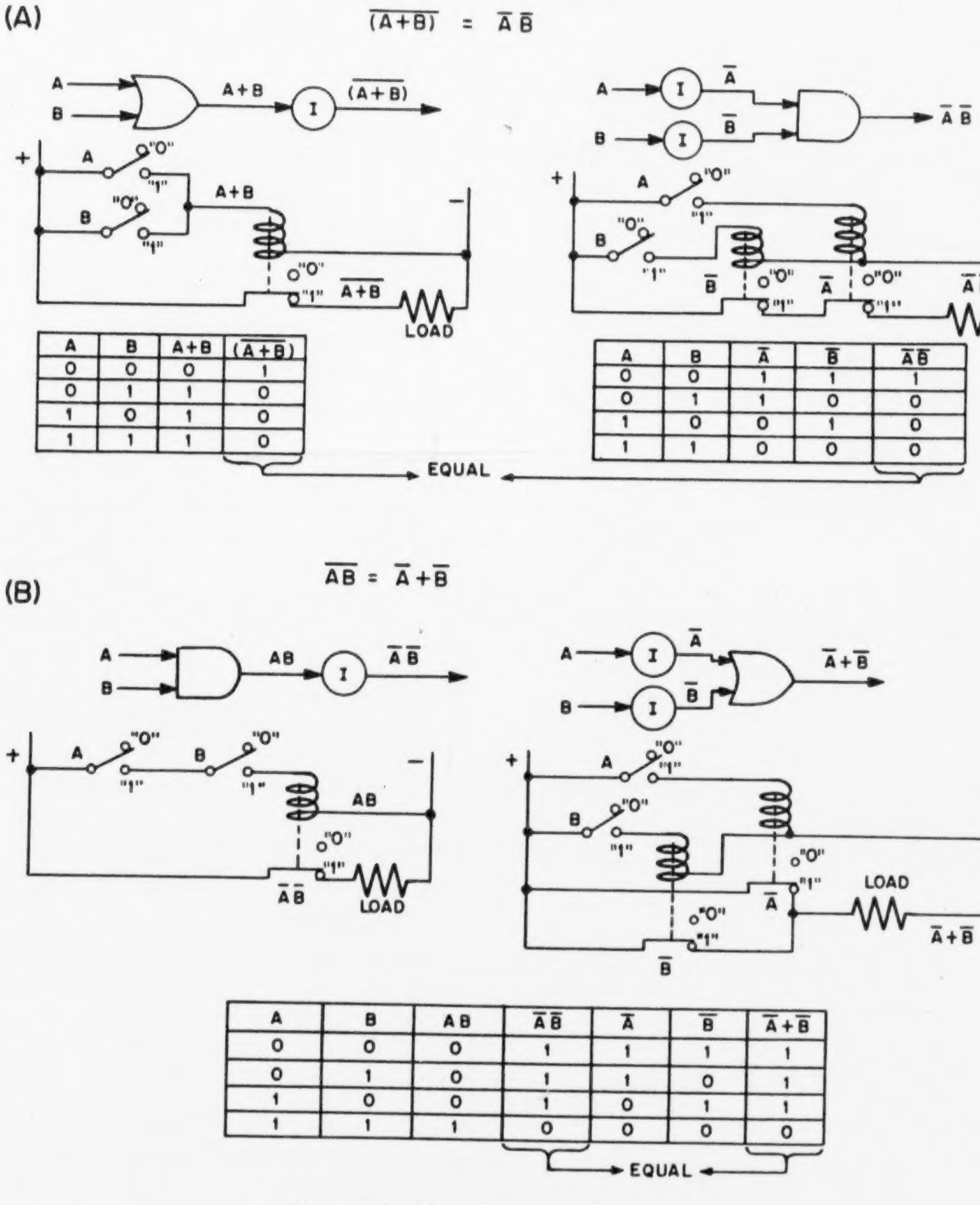
$$A\overline{BC} + \overline{D + E} = A(\overline{B} + \overline{C}) + \overline{D}\,\overline{E}$$

Notice that the grouping of letters must be maintained.

DISTRIBUTIVE LAW

There are two parts to the distributive law as shown in figure 7-22. The first identity is

$$A(B + C) = AB + AC$$


(A)
$\overline{(A+B)} = \bar{A}\,\bar{B}$
A
B
A+B
I
$\overline{(A+B)}$
$\bar{A}$
$\bar{B}$
$\bar{A}\,\bar{B}$
+
−
"0"
"1"
LOAD
EQUAL
(B)
$\overline{AB} = \bar{A}+\bar{B}$
AB
$\overline{AB}$
$\bar{A}+\bar{B}$

A	B	A+B	$\overline{(A+B)}$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

A	B	$\bar{A}$	$\bar{B}$	$\bar{A}\,\bar{B}$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

A	B	AB	$\overline{AB}$	$\bar{A}$	$\bar{B}$	$\bar{A}+\bar{B}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

Figure 7-21.—Law of Dualization.

and in order to obtain an output of 1 the A must be 1 and either B or C must be 1. This law is similar to the law of algebra which states that multiplication distributes over addition.

The second identity is

$$A + BC = (A + B)(A + C)$$

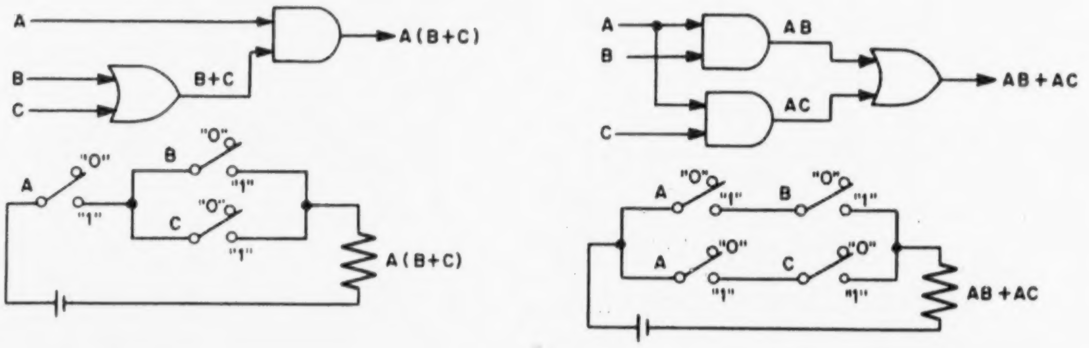
and in order to obtain an output of 1, at least one term in each of the parentheses must be 1. THIS LAW DOES NOT APPLY TO ORDINARY ALGEBRA. If this law did apply to ordinary algebra it would indicate that addition distributes over multiplication. In Boolean algebra this is true. Examples of the distributive law are as follows:

$$A(B + C + D) = AB + AC + AD$$

and

$$A + (B + C)(D + E) = (A + B + C)(A + D + E)$$

(A) $A(B+C) = (AB) + (AC)$

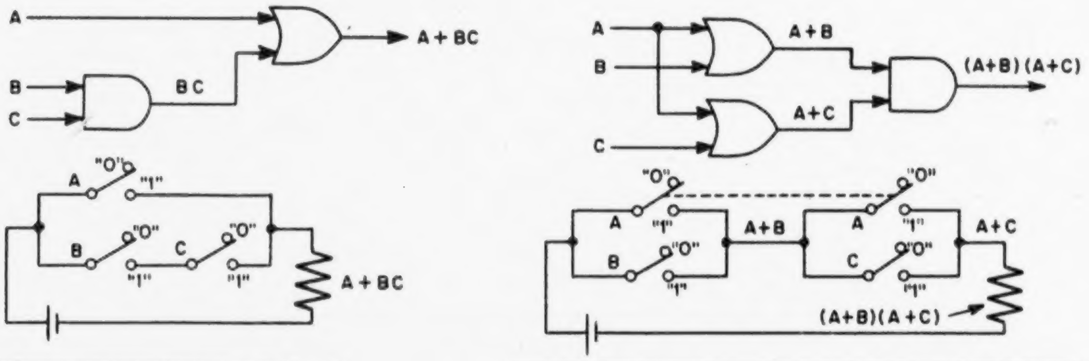

A
B
C
B+C
A(B+C)
AB
AC
AB+AC
"0"
"1"
A(B+C)
AB+AC

A	B	C	B+C	A(B+C)
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

A	B	C	AB	AC	AB+AC
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	0	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	1	1

EQUAL

(B) $A+BC = (A+B)(A+C)$


A
B
C
BC
A + BC
A+B
A+C
(A+B)(A+C)
"0"
"1"
A + BC
A+B
A+C
(A+B)(A+C)

A	B	C	BC	A+BC
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

A	B	C	A+B	A+C	(A+B)(A+C)
0	0	0	0	0	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

EQUAL

Figure 7-22.—Distributive Law.

LAW OF ABSORPTION

The law of absorption is shown in figure 7-23. This law is written as

$$A(A + B) = A$$

and

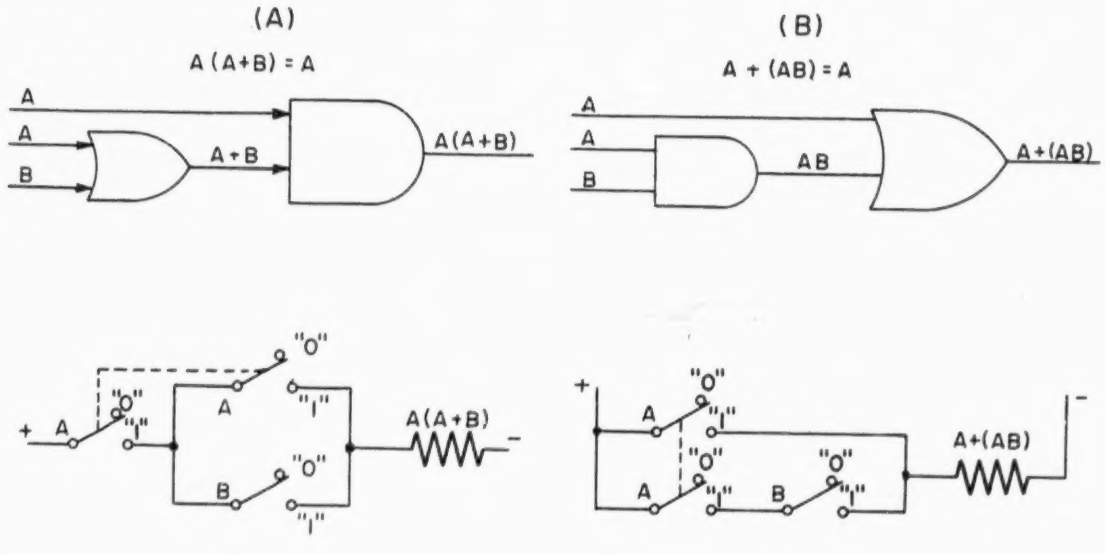
$$A + AB = A$$

and indicates that the output is 1 whenever A is 1. Examples are

$$D(1 + E) = D \cdot 1 = D$$

and

$$\begin{aligned} A + AB + AC &= A(1 + B + C) \\ &= A \cdot 1 \\ &= A \end{aligned}$$


(A)
A (A+B) = A
A
A
B
A+B
A(A+B)
"0"
"1"
+
A
A
B
A(A+B)
−
(B)
A + (AB) = A
A
A
B
AB
A+(AB)
+
"0"
"1"
A
A
B
A+(AB)
−

A	B	A + B	A(A+B)
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

EQUAL

A	B	AB	A+(AB)
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

EQUAL

Figure 7-23.—Absorption Law.

CHAPTER 8

BOOLEAN SIMPLIFICATION

This chapter will be devoted toward understanding how output expressions may be simplified by various methods. By proper application of simplification techniques the systems designer can be sure a circuit is in its simplest algebraic form. It is obvious that the more simple a circuit is the fewer components are needed and the cheaper the cost will be to construct the circuit.

It should be understood that there are cases where the simplest electronic circuit is not the result of the simplest algebraic expression; however, by application of Boolean algebra to manipulate a simplified expression, a designer can obtain a circuit that is simplest from an electronic viewpoint.

After a designer determines his Boolean function by means of the minterm or maxterm expression, he must know whether this expression may be simplified. By simplified, we mean that another expression may be determined that will represent the same function with less equipment. For example, the designer may arrive at the function $f(A,B,C) = A\overline{B} + B\overline{C} + \overline{B}C + \overline{A}B$. This function can be simplified to give $f(A,B,C) = A\overline{B} + B\overline{C} + \overline{A}C$ which may be easier to construct.

ORDER OF EXPRESSION

When describing Boolean functions, it is often necessary to identify them as to their order. The order is defined, for example, so that the cost of the logic circuit may be determined without constructing the circuit. Higher order expressions generally result in more cost to construct the circuits.

To determine the order of a Boolean expression, we must first inspect the quantity within the parentheses. If this quantity contains only an AND operation(s), or only an OR operation(s), this quantity is first order. If the quantity contains both an AND and an OR operation(s), it is considered a second-order quantity.

The next step is to consider the relationship of the quantity within the parentheses and the other variables within the brackets of the expression. Again, if the parenthesized quantity is combined with the other bracketed variables with either an AND operation(s) or an OR operation(s), the order is increased accordingly. This process is continued until the final order of the expression is obtained.

To find the order of the expression

$$[(AB + C)\,D + E]\,F + G$$

first consider the parenthesized quantity

$$(AB + C)$$

This quantity contains an AND operation and an OR operation; therefore, it is second-order. Now consider the quantity in brackets; that is,

$$[(AB + C)\,D + E]$$

The parenthesized quantity (second-order) is combined with an AND and an OR operation; therefore, the quantity in brackets is fourth-order. Finally, the quantity in brackets (fourth-order) is combined with an AND and an OR operation in

$$[(AB + C)\,D + E]\,F + G$$

and the entire expression is then sixth-order.

To find the order of the expression

$$[(AB)(CD + EF) + G]$$

begin with the parenthesized quantity which has the highest order; that is

$$(CD + EF)$$

It contains both AND and OR operations and is therefore second-order. The quantity in brackets contains a second-order quantity combined with both AND and OR operations and is a fourth-order expression.

SIMPLIFICATION

Since one input signal may accomplish the function of another, we may, in many cases, eliminate the superfluous signal. We use the basic laws of Boolean algebra in order to eliminate parts of expressions without changing the logic state of the output.

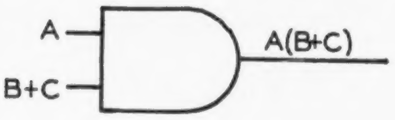
It is important to recognize whether one part of an expression is equal to another; that is, by recalling the commutative law we find that

$$ABC = BAC$$

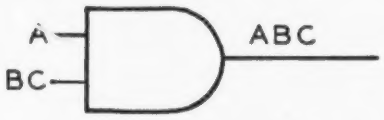
and

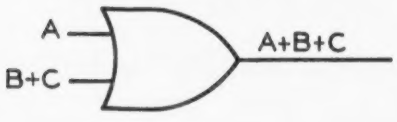
$$A(B + C) = (C + B)\,A$$

Previously, when writing output expressions in order to reproduce circuit diagrams, we use parentheses in all cases except for ANDed inputs to OR symbols. In simplification, we use parentheses only when we have an ORed input to an AND symbol. That is,

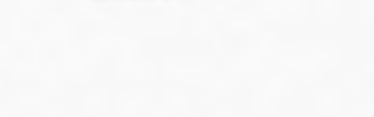


The other three cases are shown without parentheses as follows:





and



To simplify

$$(AB + C) + D + EF$$

we write

$$AB + C + D + EF$$

The idempotent law states that

$$AA = A$$

and

$$A + A = A$$

This law is used to simplify expressions as follows:

$$(AB)(AB) = AB$$

and

$$\begin{aligned} A+B+A+C+B+D &= A+A+B+B+C+D \\ &= A+B+C+D \end{aligned}$$

and

$$A\bar{B}\bar{C} + A\bar{B}\bar{C} = A\bar{B}\bar{C}$$

PROBLEMS: Simplify the following:

1. $(AA)(BB)$
2. $AB + (AB + CD)$
3. $(AB + C)(D\bar{E})(\bar{E}D)(C + BA)$
4. $(ABC + DE) + CBA$

ANSWERS:

1. AB
2. $AB + CD$
3. $(AB + C)(D\bar{E})$
4. $ABC + DE$

When simplifying expressions which contain negations, the use of the law of double negation is used. This law indicates that whenever two negation bars of equal length cover the same letter or expression, both bars may be removed without affecting the value of the expression. To simplify the expression

$$\overline{\overline{AB}}$$

we write

$$\overline{\overline{AB}} = AB$$

and the expression

$$\overline{\overline{\overline{AB}}} + \overline{\overline{C}} = \overline{AB} + C$$

Notice that we removed only two bars from above AB.

To simplify the expression

$$(\overline{\overline{A + B}} + C) + \overline{\overline{C}} + (A + B)$$

we use the laws which we have discussed to this point and write

$$\overline{\overline{A + B}} = A + B$$

and

$$\overline{\overline{C}} = C$$

therefore

$$(A + B + C) + C + (A + B)$$

equals

$$(A + B + C) + C + (A + B)$$

then by removing the parentheses and applying the commutative law write

$$A + A + B + B + C + C$$

then by the idempotent law this equals

$$A + B + C$$

PROBLEMS: Simplify the following expressions.

1. $(A\overline{\overline{BC}} + D)\ E + \overline{\overline{\overline{F}}}$
2. $(\overline{\overline{AB}}C + D)(D + \overline{\overline{BA}}C)$

ANSWERS:

1. $(ABC + D)\ E + \overline{F}$
2. $ABC + D$

In the discussion of the complementary law, the logic state of the output is considered. This law indicates that when any letter or expression is ANDed with its complement, the output is 0. When any letter or expression is ORed with its complement, the output is 1; that is,

$$A\overline{A} = 0$$

and

$$A + \overline{A} = 1$$

The logic state of

$$(A + B)\overline{(A + B)} = 0$$

and the logic state of

$$AB + \overline{AB} = 1$$

To simplify the expression

$$\overline{AB\overline{\overline{C}}} + A(\overline{C}B)$$

we write

$$\overline{AB\overline{\overline{C}}} + A(\overline{C}B)$$

$$\overline{AB}C + AB\overline{C}$$

and notice that

$$\overline{AB}C$$

is the complement of

$$AB\overline{C}$$

therefore

$$\overline{AB\overline{\overline{C}}} + A(\overline{C}B) = 1$$

PROBLEMS: Simplify the following expression.

1. $A\overline{A}\overline{\overline{A}}$
2. $(A + \overline{\overline{\overline{A}}})\ A\overline{A}$
3. $\overline{A(BC)\ D} + (AB)\overline{\overline{(CD)}}$

ANSWERS:

1. 0
2. 0
3. 1

Since intersection indicates the AND operation, we concern ourselves with the law of intersection for simplification of ANDed operations. The law of intersection is

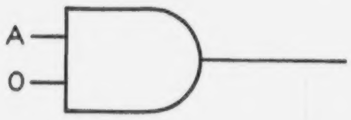
$$A \cdot 1 = A$$

and

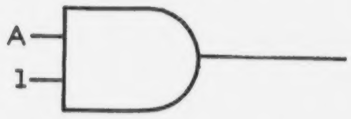
$$A \cdot 0 = 0$$

Therefore, whenever we have an input of 1 ANDed with an input A, the output will have the same binary value as A. Also, if the input 0 is ANDed with A, the output will have the binary value of 0.

This is best illustrated by the diagrams. If two inputs to an AND circuit are



the output will be 0. If the two inputs are



then the output will take the value of A; that is, if A is 1, the output is 1; and if A is 0, the output is 0.

To simplify the expression

$$(AB + C)\ 0$$

we need only consider this as straightforward multiplication and write the answers as 0.

To simplify the expression

$$(AB + C)(D + E) \cdot 1$$

we apply the same logic and write $(AB + C)(D + E)$.

When we desire to simplify an expression such as

$$(A + B)\overline{(A + B)}(C + D)$$

we use the complementary law to find

$$(A + B)\overline{(A + B)} = 0$$

and the law of intersection to find

$$0(C + D) = 0$$

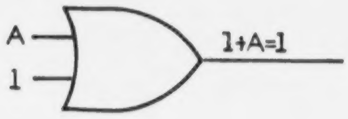
PROBLEMS: Simplify the following expressions.

1. $\overline{A}(B + \overline{B})$
2. $\overline{\overline{A}} + (B + \overline{B})A$
3. $(A + \overline{A})(BC + DE)$

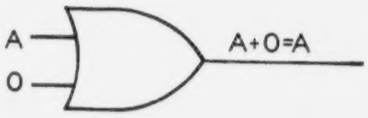
ANSWERS:

1. $\overline{A}$
2. A
3. $(BC + DE)$

The law of union is used in simplifying expressions in somewhat the same manner as the law of intersection is used. The difference between the two laws is that the law of union is considered as straightforward addition; that is,



and



To simplify the expression

$$A + B + 1$$

we write the answer as 1 because we may have only 0 or 1 as the output; and since we are using the process of addition, the values that A and B take have no effect on the output.

DeMorgan's theorem is a useful tool in simplifying Boolean algebra expressions. It is presented basically in the following two equations:

$$\overline{A + B} = \bar{A}\bar{B}$$

$$\overline{AB} = \bar{A} + \bar{B}$$

These equations indicate that in the process of simplification whenever you split or join vincula you change the sign; that is, AND to OR, or OR to AND.

When applying DeMorgan's theorem to the expression

$$\overline{A + B + C}$$

write

$$\bar{A}\ \bar{B}\ \bar{C}$$

Further examples are:

$$\bar{A} + \bar{B} + \bar{C} = \overline{ABC}$$

and

$$\bar{A}\bar{B}\bar{C} = \overline{A + B + C}$$

It should be noted that when you change signs in an expression you must group the same letters that were originally grouped; that is,

$$\overline{AB + C} = (\bar{A} + \bar{B})\bar{C}$$

Other examples are:

$$(\bar{A} + \bar{B})\bar{C} = \overline{AB + C}$$

and

$$(\bar{A} + \bar{B})\ \bar{C} + \bar{D} = \overline{(AB + C)\ D}$$

In cases where the vinculum covers part of an expression, the signs under the vinculum change while the signs outside the vinculum do not change; that is,

$$A + \overline{B(C + D) + E} = A + (\bar{B} + \bar{C}\bar{D})\bar{E}$$

and

$$\overline{(AB + C)D} + \overline{EF} = (\bar{A} + \bar{B})\bar{C} + \bar{D} + \bar{E} + \bar{F}$$

Using DeMorgan's theorem to simplify

$$AB + CD + \bar{A} + \bar{B}$$

we write

$$\bar{A} + \bar{B} = \overline{AB}$$

then substituting we have

$$AB + CD + \overline{AB}$$

and

$$AB + \overline{AB} = 1$$

therefore

$$AB + CD + \overline{AB}$$

$$= 1 + CD$$

$$= 1$$

We may also split a vinculum to simplify an expression as follows:

$$A + B + CD + \overline{AB}$$

$$= A + B + CD + \bar{A} + \bar{B}$$

$$= A + \bar{A} + B + \bar{B} + CD$$

$$= 1 + 1 + CD$$

$$= 1$$

In some cases it may be necessary to manipulate one part of an expression so that it is the complement of another part of the expression. This may be accomplished in the following manner.

If, when simplifying the expression

$$[A + \overline{B(C + D)}][\bar{A}B(C + D)] + E$$

we choose to split the vinculum in the first parentheses, we have

$$A + \bar{B} + \bar{C}\bar{D}$$

which is not in a more simple form. Therefore, we elect to add two vincula over A and have

$$\bar{\bar{A}} + \overline{B(C + D)}$$

then we join the vincula to find

$$\overline{\overline{A}\;B(C + D)}$$

which is the complement of the expression in the second parentheses. Then,

$$[\overline{\overline{A}\;B(C+D)}][\overline{A}\;B(C + C)] = 0$$

and

$$0 + E = E$$

When simplifying expressions involving vincula, it must be understood that if any letter in an expression has more than one vinculum over it, the expression is not in the simplest form; that is, the expression

$$\overline{\overline{A} + BC}$$

is not in the simplest form. In this case we would split the long vinculum and have

$$\overline{\overline{A}}\;(\overline{B} + \overline{C})$$

which simplifies to

$$A\;(\overline{B} + \overline{C})$$

In order to simplify the expression

$$\overline{\overline{\overline{(A + B)}\;C} + \overline{(D + \overline{E})\;F}}$$

by the methods discussed would require many steps. An easier method is to remember that if one vinculum is removed the operation changes, and if two vincula are removed the operation remains the same; that is,

$$\overline{\cdot} = +$$

$$\overline{\overline{\cdot}} = \cdot$$

$$\overline{+} = \cdot$$

$$\overline{\overline{+}} = +$$

$$\overline{\overline{\overline{+}}} = \cdot$$

Using this technique to simplify the expression

$$\overline{A\;\overline{(B + C)}}$$

we think

$$\overline{A}\;\overline{\cdot}\;(\overline{\overline{B}}\;\overline{\overline{+}}\;\overline{\overline{C}})$$

and write

$$\overline{A} + B + C$$

To simplify the expression

$$\overline{A\;\overline{B}(C + \overline{DE}) + F}$$

we think

$$\overline{A}\,\overline{\cdot}\,\overline{\overline{B}}\,\overline{\cdot}\,[\overline{C}\;\overline{+}\;\overline{\overline{D}}\,\overline{\overline{\cdot}}\,\overline{\overline{E}}]\;\overline{+}\;\overline{F}$$

and write

$$[\overline{A} + B + [\overline{C}\cdot(D\cdot E)]]\cdot\overline{F}$$

and then

$$(\overline{A} + B + \overline{C}\;D\;E)\;\overline{F}$$

We used the double brackets and parentheses in order to maintain the proper groupings.

PROBLEMS: Simplify the following expression.

1. $\overline{(A + \overline{A})(\overline{BC} + \overline{C} + \overline{B})}$
2. $\overline{\overline{(AB + C)}(\overline{A} + \overline{B})\;\overline{C}}$

ANSWERS:

1. BC
2. AB + C

To this point in our discussion we have determined there are two ways in which DeMorgan's theorem may be applied in simplifying expressions. First, we may join vincula to form the complement of a term or expression. Second, we may split vincula and then apply other laws to further simplify the expression.

In many cases an expression may not be simplified in one form while simplification of the expression may be accomplished if the expression is in another form. The distributive law is one which enables us to change the form of an expression. The distributive law contains two identities. These are:

$$A\;(B + C) = AB + AC$$

and

$$A + BC = (A + B)(A + C)$$

Notice that the second identity is true for Boolean algebra but is not true for ordinary algebra. The following conversions of expressions from one form to another form is given for understanding the distributive law.

$$A\ (B + C + D) = AB + AC + AD$$

$$AB\ (C + D) = ABC + ABD$$

and

$$A + BC = (A + B)(A + C)$$

$$A + BCD = (A + B)(A + C)(A + D)$$

Notice in the conversion that either side of the equations may be changed to the other form. That is,

$$ABC + ABD = AB\ (C + D)$$

and

$$(A + B + C)(A + E + F) = A + (B + C)(E + F)$$

In ordinary algebra the first of these equations was called carryout the distributive law and the last two equations were called factoring.

The expression

$$AB + A\overline{B}$$

cannot be simplified in its present form but by application of the distributive law we have

$$\begin{aligned} AB + A\overline{B} &= A\ (B + \overline{B}) \\ &= A \cdot 1 \\ &= A \end{aligned}$$

Also in this category is

$$\begin{aligned} (A + \overline{B})(A + B) &= A + (\overline{B}\ B) \\ &= A + 0 \\ &= A \end{aligned}$$

PROBLEMS: Convert the form of each of the following expressions to another form.

1. $AB + CDE$
2. $ABC + ABD + ACD$
3. $(A + B)(A + D)$
4. $(A + BC)(BC + D + E)$

ANSWERS:

1. $(AB + C)(AB + D)(AB + E)$
2. $A\ (BC + BD + CD)$
3. $A + BD$
4. $BC + A\ (D + E)$

In the previous discussion of the distributive law, the change of the form of the expressions is used to aid simplification. The final determination of which form to use is made by finding which form requires fewer gates. There are no hard-and-fast rules which may be used to determine which form is more simple.

For our discussion we will consider the more simple form of an expression as having no vinculum extending over more than one letter and having no parentheses.

PROBLEMS: Simplify the following expressions.

1. $AB + \overline{AB}C$
2. $ABC + AB\overline{C}$
3. $AB\ (AE + \overline{B} + C\overline{A})$

ANSWERS:

1. $AB + C$
2. AB
3. ABE

The law of absorption is very closely related to the distributive law in that we use the distributive law to show the law of absorption. The law of absorption is shown by

$$A + AB = A$$

and

$$A\ (A + B) = A$$

If we take

$$A + AB$$

and factor out an A we have

$$A\ (1 + B)$$
$$= A \cdot 1$$
$$= A$$

and also

$$A\ (A + B)$$
$$= AA + AB$$
$$= A + AB$$

where we used the distributive law.

Actually, the law of absorption eliminates terms of an expression which are not needed. This may be seen in the following.

In the expression

$$ABC + AB$$

if A and B each have the value of 1, then the output value is 1 regardless of the value of the term ABC. Therefore, the term ABC is not necessary because the only values which will make ABC equal 1 is for A, B, and C to each have the value of one and we have already agreed the output is 1 if A and B are equal to 1.

When we use the law of absorption to simplify this expression, we have

$$ABC + AB$$
$$= AB\ (C + 1)$$
$$= AB$$

PROBLEMS: Simplify the following expressions.

1. AB + ABC + ABCD
2. AB + ABC + A
3. AB + CD + ABE

ANSWERS:

1. AB
2. A
3. AB + CD

The law of absorption is also used to simplify expressions of the type

$$\overline{\overline{A}}\ \overline{\overline{A}\ B}$$

To simplify this expression we write

$$\overline{\overline{A}}\ \overline{\overline{A}\ B} = \overline{\overline{A}}\ (\overline{\overline{A}} + \overline{B})$$
$$= A\ (A + \overline{B})$$
$$= A + A\ \overline{B}$$
$$= A\ (1 + \overline{B})$$
$$= A$$

Also, to simplify

$$A\ (B + C + \overline{\overline{A} + \overline{D}})\ D$$

we write

$$A\ (B + C + \overline{\overline{A} + \overline{\overline{D}}})\ D$$
$$= A\ (B + C + AD)\ D$$
$$= A\ D(B + C + AD)$$
$$= ABD + ABC + AD$$
$$= AD\ (B + C + 1)$$
$$= AD \cdot 1$$
$$= AD$$

PROBLEMS: Using any or all of the basic laws discussed, simplify the following expressions.

1. $AB + CD\overline{D} + \overline{\overline{BA}}$
2. $A + B + AC\overline{A}C$
3. $(\overline{B}B + AA)\ C$
4. $ABC\ (\overline{B} + B)$

ANSWERS:

1. AB
2. A + B
3. AC
4. ABC

VEITCH DIAGRAMS

There are many cases in which simplification of an expression is so involved that it becomes impractical to attempt the simplification by algebraic means. This situation may be averted by the use of the Veitch diagram. Veitch diagrams provide a very quick and easy way for finding the simplest logical equation needed to express a given function or expression. A Veitch diagram is a block of squares on which you plot an expression.

As previously mentioned, we will consider an expression as being in simplified form when no vinculum extends over more than one letter and the expression contains no parentheses. This process results in an expression in minterm form. Recall that a minterm is the symbolic product of a given number of variables; that is,

$$ABC$$

is a minterm of three variables and

$$ABCD$$

is a minterm of four variables.

An expression is in minterm form if it is composed only of minterms connected by the operation OR sign. An example of a minterm form expression is

$$AB + C + \overline{D}EF$$

while

$$AB + C + D(A + C)$$

is not in minterm form.

In order to place any expression in minterm form we need only split or remove vincula, remove parentheses, and simplify within the term. When the expression is in minterm form, further simplification is unnecessary if Veitch diagrams are to be used; that is, to convert the expression

$$\overline{A + B} + RS$$

to minterm forms, we write

$$\overline{A + B} + RS$$

$$= \overline{A}\overline{B} + RS$$

which is in minterm form.

PROBLEMS: Convert the following expressions to minterm form.

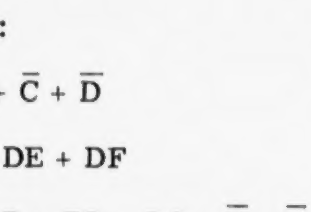
1. $A + B + \overline{CD}$
2. $\overline{A}\overline{B}C + D(E + F)$
3. $\overline{\overline{AB}} + CD + \overline{\overline{E}}B + \overline{\overline{BC}D} + \overline{\overline{\overline{E}}}$
4. $(AB + C)B + D(E + \overline{D})$

ANSWERS:

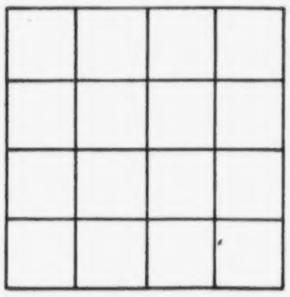
1. $A + B + \overline{C} + \overline{D}$
2. $\overline{A}\overline{B}C + DE + DF$
3. $AB + CD + EB + BC + \overline{D} + \overline{E}$
4. $AB + BC + DE$

In order to form a Veitch diagram it is necessary to know the number of possible minterms which may be formed from the variables in the expression to be simplified. To determine the number of minterms, raise 2 to the power of the number of variables; that is, if we have three variables, then we have 2^3 minterms possible.

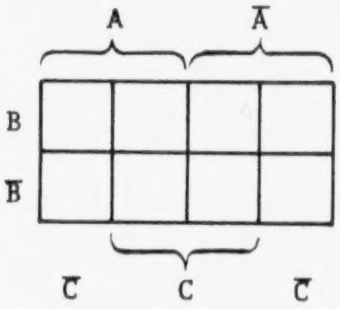
When we construct a Veitch diagram, there is one square for each minterm. A three-variable Veitch diagram is drawn as



and a four-variable Veitch diagram is drawn as

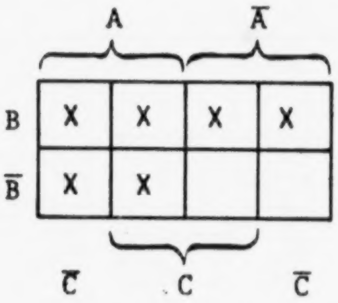


We label a Veitch diagram of eight squares as


A
$\overline{A}$
B
$\overline{B}$
$\overline{C}$
C
$\overline{C}$

Notice that half of the squares are assigned to each variable and the other half of the squares are assigned to the complements of the variables. Also, each variable overlaps every other variable and every complement except its own.

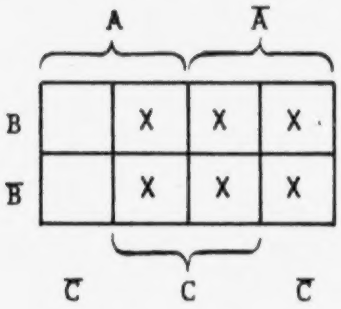
When we plot an expression such as A + B, we place an X in every square that is A and in every square that is B; that is,


A
$\overline{A}$
B
X
X
X
X
$\overline{B}$
X
X
$\overline{C}$
C
$\overline{C}$

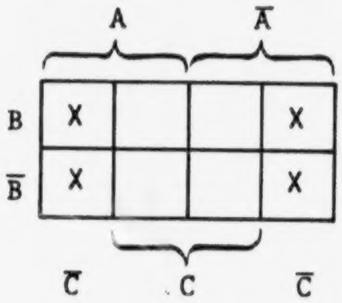
If we desire to plot the expression

$$\overline{A} + C$$

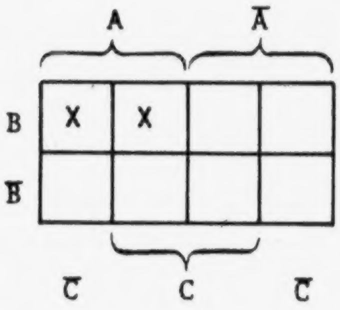
we write


A
$\overline{A}$
B
X
X
X
$\overline{B}$
X
X
X
$\overline{C}$
C
$\overline{C}$

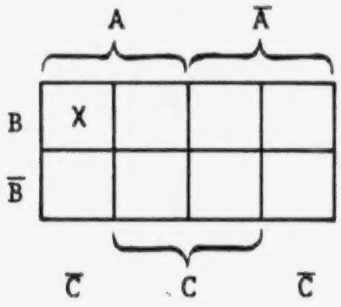
and to plot $\overline{C}$ we write


A
$\overline{A}$
B
X
X
$\overline{B}$
X
X
$\overline{C}$
C
$\overline{C}$

When we plot a term such as AB, we plot only the squares common to both A and B; that is,


A
$\overline{A}$
B
X
X
$\overline{B}$
$\overline{C}$
C
$\overline{C}$

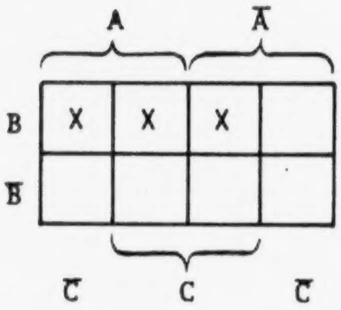
and when we plot a term such as $AB\overline{C}$, we write an X in the squares common to all those variables; that is,


A
$\bar{A}$
B
$\bar{B}$
X
$\bar{C}$
C
$\bar{C}$

To plot an entire expression we plot each term on the same diagram. To plot the expression

$$AB + CB + \bar{A}BC$$

we write

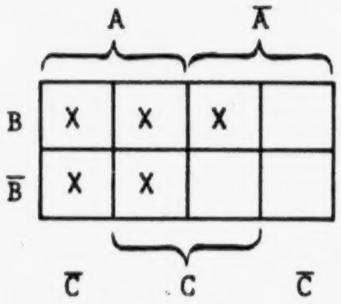

A
$\bar{A}$
B
$\bar{B}$
X
X
X
$\bar{C}$
C
$\bar{C}$

PROBLEMS: Plot the following expressions on Veitch diagrams.

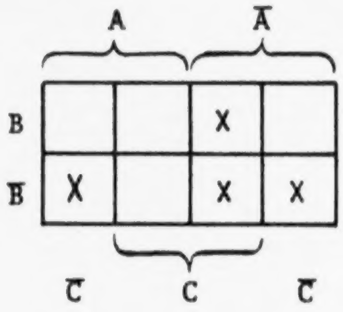
1. $BC + A$
2. $\bar{A}C + \bar{B}\bar{C}$
3. $\bar{C}B + A\bar{C} + \bar{B}C + \bar{B}\bar{A}$

ANSWERS:

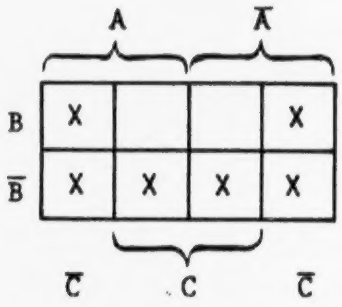
1.


A
$\bar{A}$
B
$\bar{B}$
X
X
X
X
X
$\bar{C}$
C
$\bar{C}$

2.


A
$\bar{A}$
B
$\bar{B}$
X
X
X
X
$\bar{C}$
C
$\bar{C}$

3.


A
$\bar{A}$
B
$\bar{B}$
X
X
X
X
X
X
$\bar{C}$
C
$\bar{C}$

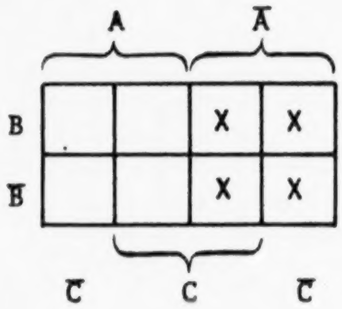
It should be noted that when plotting squares common to variables of a minterm a single variable term occupies four squares, a two-variable term occupies two squares, and a three-variable term occupies one square.

To extract the simplest expression from a Veitch diagram we look for, in order, four plotted squares which may be described by a one-variable term, two plotted squares which may be described by a two-variable term, and then one plotted square described by a three-variable term.

EXAMPLE: Simplify the following expression by use of a Veitch diagram.

$$\bar{A}BC + \bar{A}B\bar{C} + \bar{A}\bar{B}$$

SOLUTION: First we plot the terms of the expression or a Veitch diagram by writing


A
$\bar{A}$
B
$\bar{B}$
X
X
X
X
$\bar{C}$
C
$\bar{C}$

then we follow the previous instructions and find the single variable term which expresses the plotted squares in as few terms as possible to be $\overline{A}$. Therefore, the expression

$$\overline{A}BC + \overline{A}B\overline{C} + \overline{A}\,\overline{B}$$

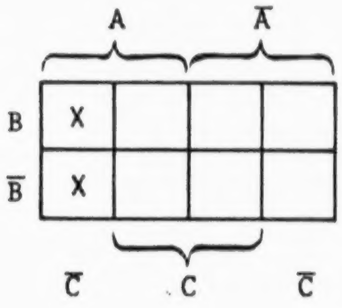
when simplified is

$$\overline{A}$$

EXAMPLE: Simplify by Veitch diagram the expression

$$AB\overline{C} + A\overline{B}\,\overline{C}$$

SOLUTION: Plot the diagram by writing



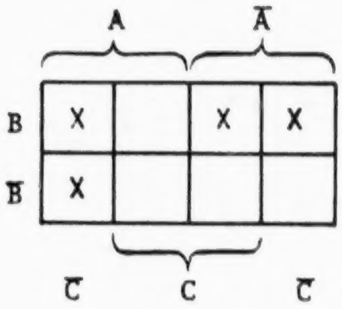
	A	A	$\overline{A}$	$\overline{A}$
B	X			
$\overline{B}$	X			
	$\overline{C}$	C	C	$\overline{C}$

then express the two plotted squares by the fewest terms possible which is $A\overline{C}$.

EXAMPLE: Use the Veitch diagram to simplify the expression

$$AB\overline{C} + A\overline{B}\,\overline{C} + \overline{A}BC + \overline{A}B\overline{C}$$

SOLUTION: Write



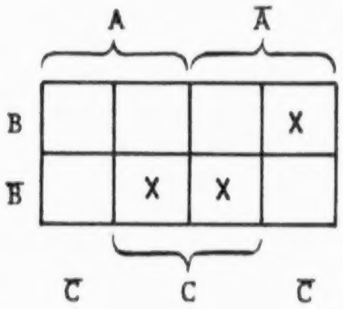
	A	A	$\overline{A}$	$\overline{A}$
B	X		X	X
$\overline{B}$	X			
	$\overline{C}$	C	C	$\overline{C}$

and the description of the plots in the fewest terms possible is

$$A\overline{C} + \overline{A}B$$

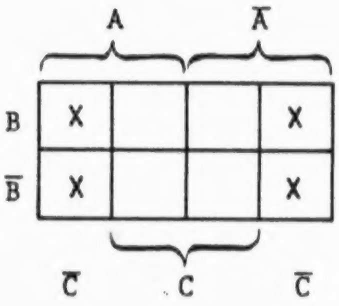
PROBLEMS: Describe the following Veitch diagrams in as few terms as possible.

1.



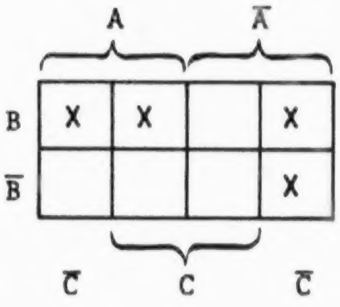
	A	A	$\overline{A}$	$\overline{A}$
B				X
$\overline{B}$		X	X	
	$\overline{C}$	C	C	$\overline{C}$

2.



	A	A	$\overline{A}$	$\overline{A}$
B	X			X
$\overline{B}$	X			X
	$\overline{C}$	C	C	$\overline{C}$

3.



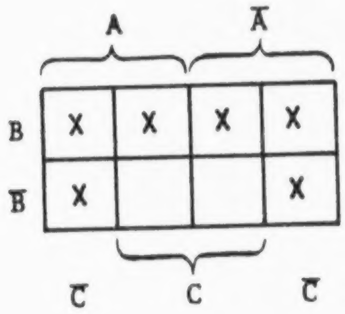
	A	A	$\overline{A}$	$\overline{A}$
B	X	X		X
$\overline{B}$				X
	$\overline{C}$	C	C	$\overline{C}$

ANSWERS:

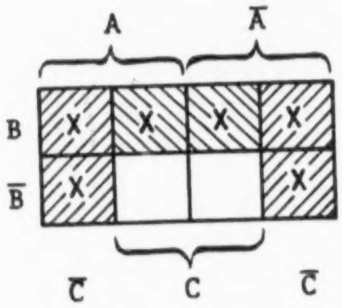
1. $\overline{B}C + \overline{A}B\overline{C}$
2. $\overline{C}$
3. $AB + \overline{A}\,\overline{C}$

When describing the plotted Veitch diagram, you should determine whether any plotted squares could be described twice. If so, this will, in many cases, result in the simplest description.

EXAMPLE: Describe the following Veitch diagram in as few variables in each term as possible.


A
$\overline{A}$
B
$\overline{B}$
X X X X
X X
$\overline{C}$
C
$\overline{C}$

SOLUTION: We could consider the plots as shown


A
$\overline{A}$
B
$\overline{B}$
X X X X
X X
$\overline{C}$
C
$\overline{C}$

and write

////// = $\overline{C}$

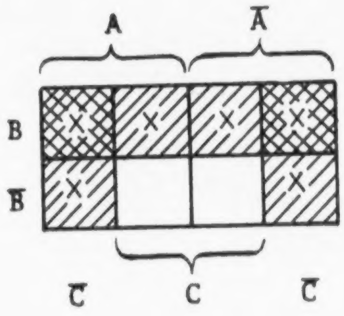
and

\\\\\\ = BC

then the description would be

$$\overline{C} + BC$$

If we could consider the plots as


A
$\overline{A}$
B
$\overline{B}$
X X X X
X X
$\overline{C}$
C
$\overline{C}$

where

////// = B

and

\\\\\\ = $\overline{C}$

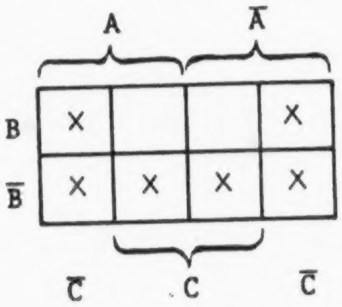
then the description would be

$$B + \overline{C}$$

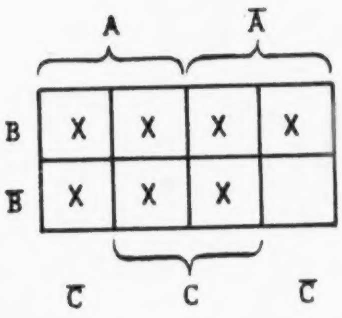
which is more simple than $\overline{C}$ + BC because of fewer variables in one term.

PROBLEMS: Describe the following diagrams by the simplest expression possible.

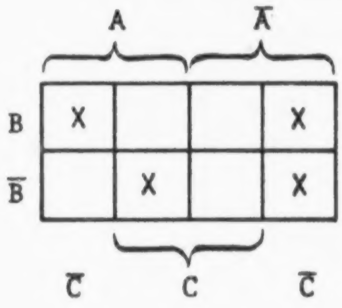
1.


A
$\overline{A}$
B
$\overline{B}$
X X
X X X X
$\overline{C}$
C
$\overline{C}$

2.


A
$\overline{A}$
B
$\overline{B}$
X X X X
X X X
$\overline{C}$
C
$\overline{C}$

3.



ANSWERS:

1. $\bar{B} + \bar{C}$

2. $A + B + C$

3. $\bar{A}\bar{C} + B\bar{C} + A\bar{B}C$

At this point it should be obvious that in order to simplify a Boolean expression by use of the Veitch diagram process it is necessary to proceed as follows:

1. Write the expression in minterm form.

2. Plot a Veitch diagram for the variables involved.

3. Extract the simplest expression from the diagram.

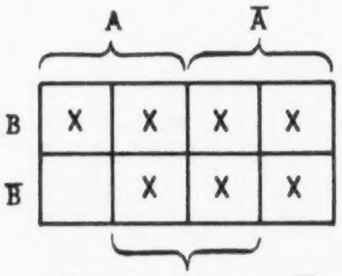
EXAMPLE: Simplify the following expression by use of a Veitch diagram.

$$AB + C + \bar{C}(\bar{A}B + \bar{A}\,\bar{B})$$

SOLUTION: Employ step (1) and write

$$AB + C + \bar{C}(\bar{A}B + \bar{A}\,\bar{B}) = AB + C + \bar{A}B\bar{C} + \bar{A}\bar{B}\bar{C}$$

Follow step (2) and write



then extract the simplest expression from the plotted Veitch diagram to find

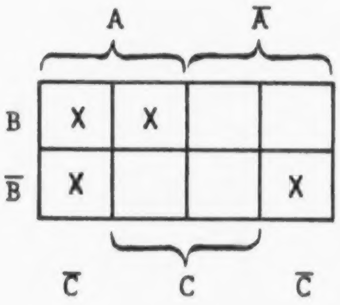
$$\bar{A} + B + C$$

EXAMPLE: Simplify the expression

$$AB + \bar{B}\,\bar{C} + A\bar{C}$$

by use of the Veitch diagram

SOLUTION: Write

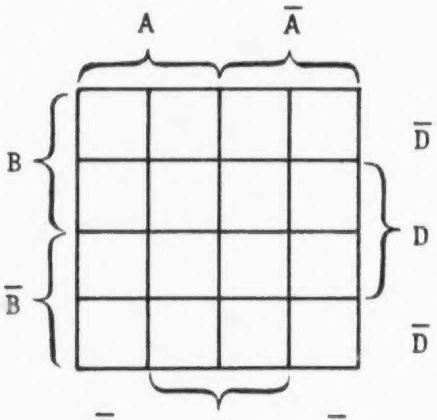


then extract the expression

$$\bar{B}\,\bar{C} + AB$$

To understand the power of the Veitch diagram method of simplification, the reader should attempt the simplification of $AB + \bar{B}\,\bar{C} + A\bar{C}$ by the use of the laws of Boolean algebra.

In the event that an expression contains four variables, we must determine the number of squares of the Veitch diagram by using 2^n where n is the number of variables. This results in 16 squares. We label the Veitch diagram as

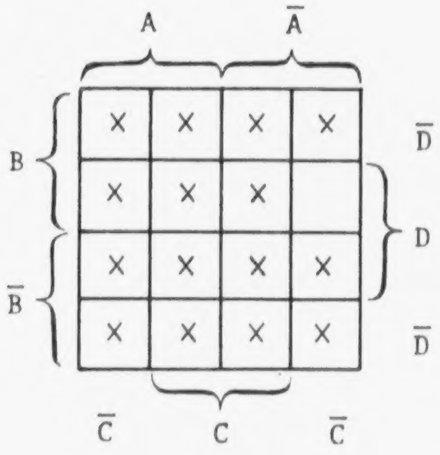


The same process is followed with 16 squares as was followed with 8 squares. The difference is that with 16 squares a single variable has 8 squares assigned, a two-variable term has 4 squares assigned, a three-variable term has 2 squares assigned, and a four-variable term is described by a single square.

To plot the expression

$$A + \bar{B} + C + \bar{D}$$

we write

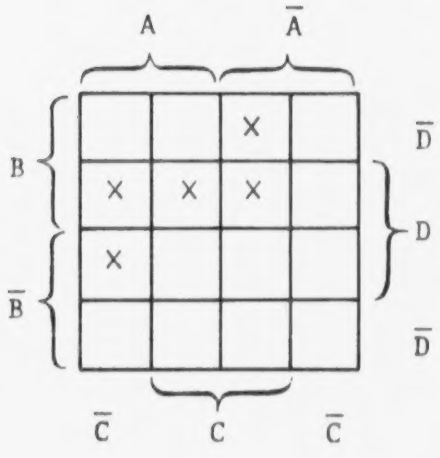


	A	A	$\bar{A}$	$\bar{A}$	
B	X	X	X	X	$\bar{D}$
B	X	X	X		D
$\bar{B}$	X	X	X	X	D
$\bar{B}$	X	X	X	X	$\bar{D}$
	$\bar{C}$	C	C	$\bar{C}$	

and to plot the expression

$$\bar{A}BC + BCD + A\bar{C}D$$

we write



	A	A	$\bar{A}$	$\bar{A}$	
B			X		$\bar{D}$
B	X	X	X		D
$\bar{B}$	X				D
$\bar{B}$					$\bar{D}$
	$\bar{C}$	C	C	$\bar{C}$	

PROBLEMS: Plot the following expressions on a Veitch diagram.

1. $B + D$

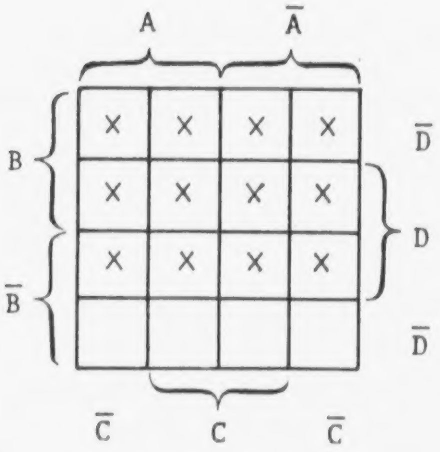
2. $\bar{A}B + \bar{C}\bar{D}$

3. $AB\bar{D} + B\bar{C}\bar{D} + ACD$

4. $\bar{A}\bar{B}CD + AB\bar{C}\bar{D} + ABCD$

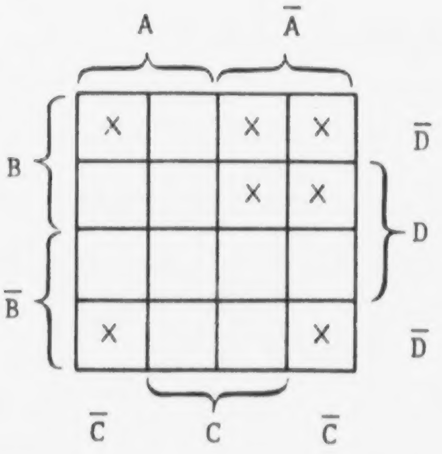
ANSWERS:

1.



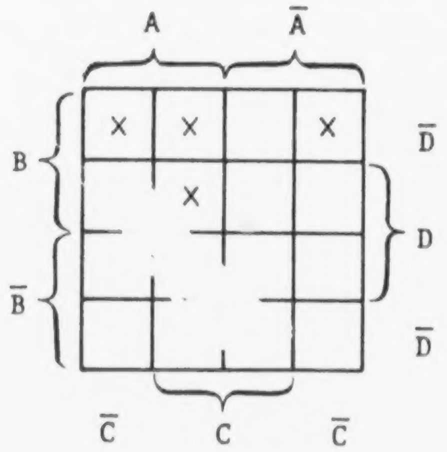
	A	A	$\bar{A}$	$\bar{A}$	
B	X	X	X	X	$\bar{D}$
B	X	X	X	X	D
$\bar{B}$	X	X	X	X	D
$\bar{B}$					$\bar{D}$
	$\bar{C}$	C	C	$\bar{C}$	

2.

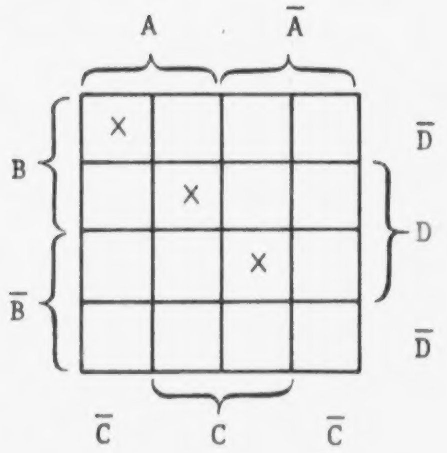


	A	A	$\bar{A}$	$\bar{A}$	
B	X		X	X	$\bar{D}$
B			X	X	D
$\bar{B}$					D
$\bar{B}$	X			X	$\bar{D}$
	$\bar{C}$	C	C	$\bar{C}$	

3.


A
$\overline{A}$
B
$\overline{B}$
$\overline{D}$
D
$\overline{D}$
X
X
X
X
$\overline{C}$
C
$\overline{C}$

4.


A
$\overline{A}$
B
$\overline{B}$
$\overline{D}$
D
$\overline{D}$
X
X
X
$\overline{C}$
C
$\overline{C}$

When extracting the simplest expression from a 16 square Veitch diagram, we use the same principles that we used on an 8 square diagram. The difference is that we now look for the following:

1. Eight plotted squares described by a one-variable term.

2. Four plotted squares described by a two-variable term.

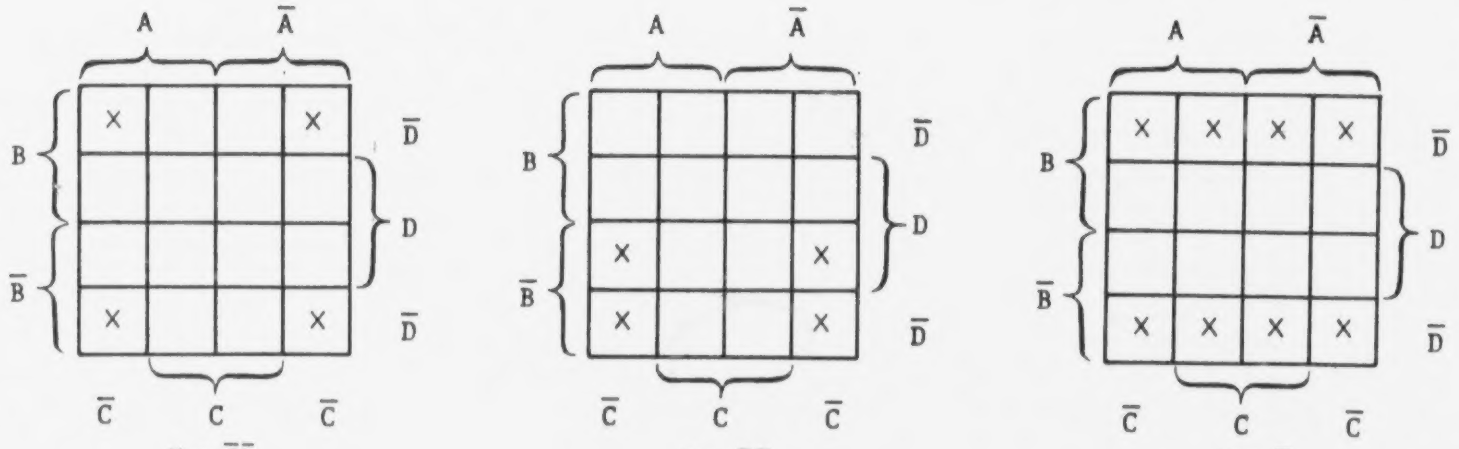
3. Two plotted squares described by a three-variable term.

4. One plotted square described by a four-variable term.

The following are examples of Veitch diagrams to illustrate patterns which should be recognized. Generally these patterns are formed by either adjacent squares or squares on the opposite ends of rows or columns.

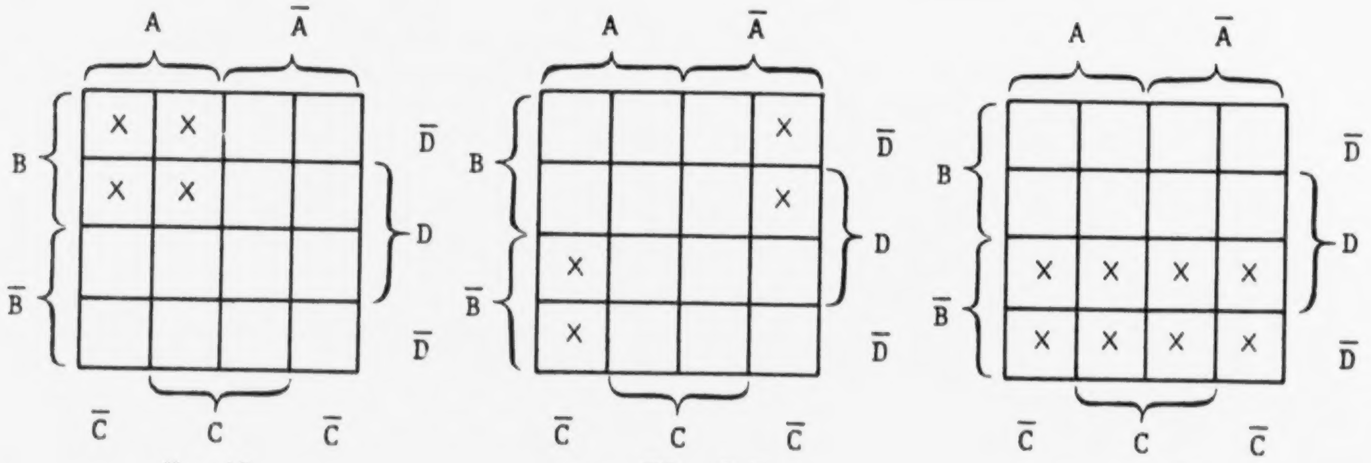
Examples of squares at opposite ends of rows or columns are shown in figure 8-1.

Examples of adjacent squares are shown in figure 8-2.


A
$\overline{A}$
B
$\overline{B}$
$\overline{D}$
D
$\overline{D}$
X
X
X
X
$\overline{C}$
C
$\overline{C}$
A
$\overline{A}$
B
$\overline{B}$
$\overline{D}$
D
$\overline{D}$
X
X
X
X
$\overline{C}$
C
$\overline{C}$
A
$\overline{A}$
B
$\overline{B}$
$\overline{D}$
D
$\overline{D}$
X
X
X
X
X
X
X
X
$\overline{C}$
C
$\overline{C}$

X = $\overline{C}\overline{D}$ X = $\overline{B}\overline{C}$ X = $\overline{D}$

Figure 8-1.—Squares at opposite ends of rows or columns.

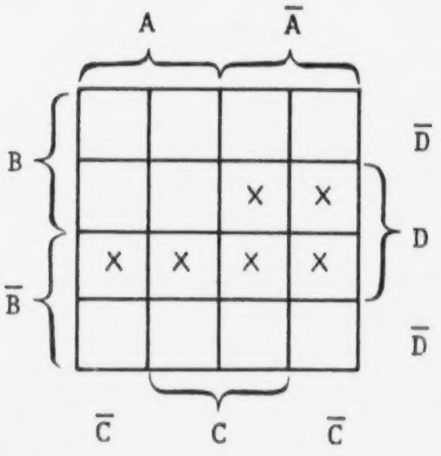

A
$\bar{A}$
B
$\bar{B}$
$\bar{D}$
D
$\bar{C}$
C
X

$X = AB$ $X = A\bar{B}\bar{C} + \bar{A}B\bar{C}$ $X = \bar{B}$

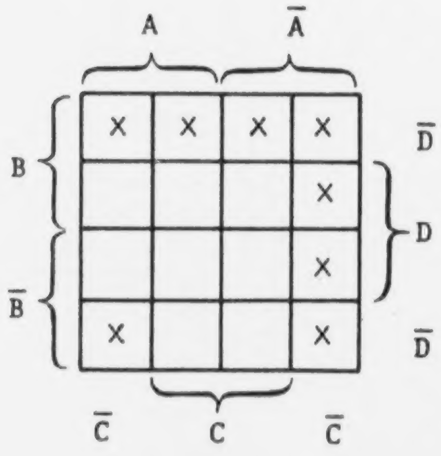
Figure 8-2.—Adjacent squares.

PROBLEMS: Describe the following plots as simply as possible.

1.


A
$\bar{A}$
B
$\bar{B}$
$\bar{D}$
D
$\bar{C}$
C
X

2.

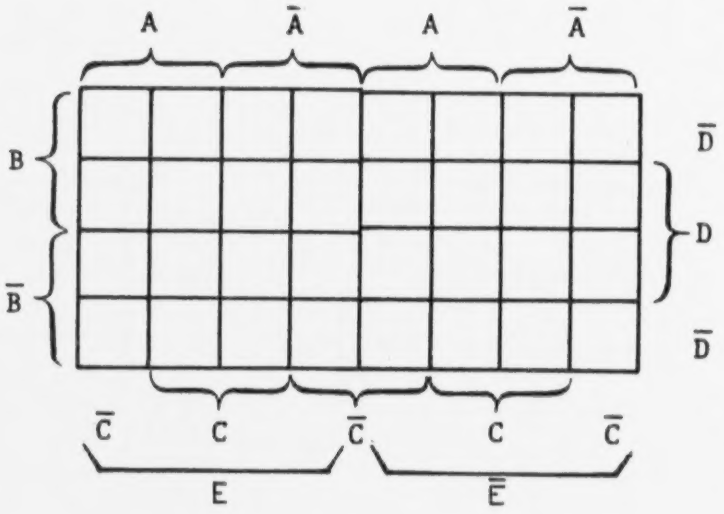

A
$\bar{A}$
B
$\bar{B}$
$\bar{D}$
D
$\bar{C}$
C
X

ANSWERS:

1. $\bar{B}D + \bar{A}D$

2. $\bar{D}\bar{C} + B\bar{D} + \bar{A}\bar{C}$

When we are faced with a five-variable expression, we use 2^5 squares. We label the Veitch diagram as

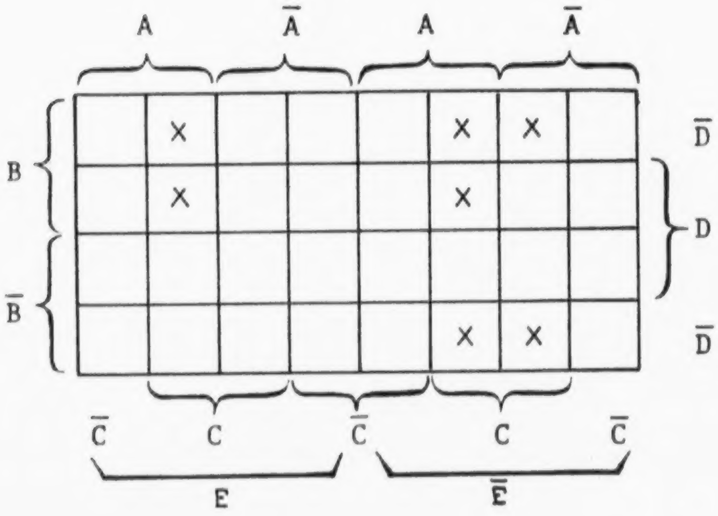

A
$\bar{A}$
A
$\bar{A}$
B
$\bar{B}$
$\bar{D}$
D
$\bar{D}$
$\bar{C}$
C
$\bar{C}$
C
$\bar{C}$
E
$\bar{E}$

In the 32 square diagram we find a one-variable term described by 16 squares, a two-variable term described by 8 squares, a three-variable term by 4 squares, a four-variable term by 2 squares, and a five-variable term by 1 square.

To plot the expression

$$A\,B\,C + C\,\overline{D}\,\overline{E}$$

we write


A
$\overline{A}$
A
$\overline{A}$
B
$\overline{B}$
$\overline{D}$
D
$\overline{D}$
X
X
X
X
X
X
X
$\overline{C}$
C
$\overline{C}$
C
$\overline{C}$
E
$\overline{E}$

In order to simplify an expression using a Veitch diagram, we follow the same procedure as before; that is, to simplify the expression

$$E(AC + B\overline{C}) + \overline{B + C + D + E}$$

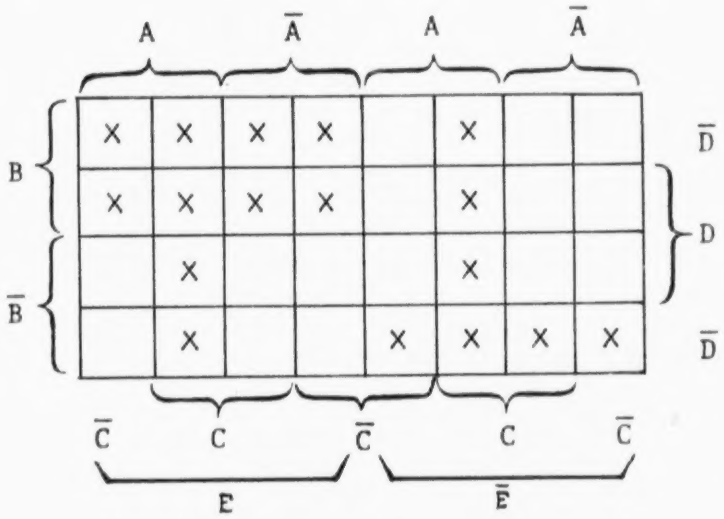
$$+\ C(BE + A\overline{B} + \overline{A + B + D + E}) + \overline{\overline{A} + \overline{B} + \overline{C}}$$

we use the laws of Boolean algebra to write the minterm expression

$$EAC + EB\overline{C} + \overline{B}\,\overline{C}\,\overline{D}\,\overline{E} + CBE$$

$$+\ CA\overline{B} + C\overline{A}\,\overline{B}\,\overline{D}\,\overline{E} + ABC$$

then plot the Veitch diagram as


A
$\overline{A}$
A
$\overline{A}$
B
$\overline{B}$
$\overline{D}$
D
$\overline{D}$
X
X
X
X
X
X
X
X
X
X
X
X
X
X
X
X
X
X
X
$\overline{C}$
C
$\overline{C}$
C
$\overline{C}$
E
$\overline{E}$

We then extract the simplest expression, as

$$BE + AC + \overline{B}\,\overline{D}\,\overline{E}$$

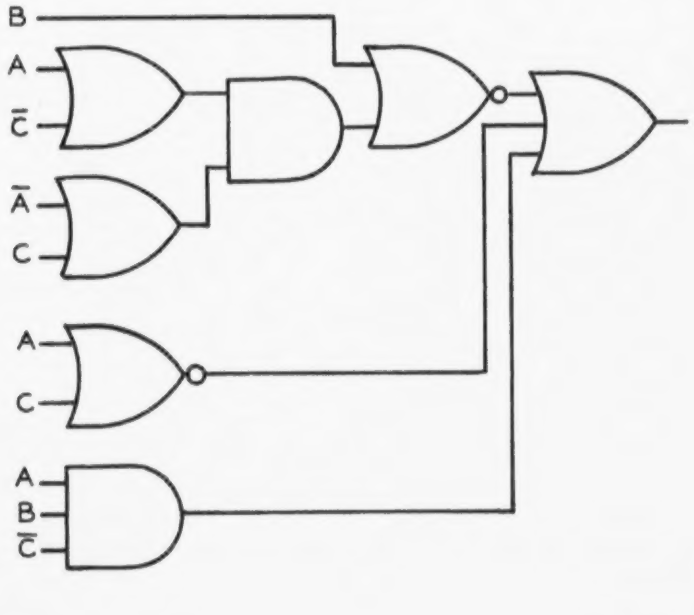
BOOLEAN EXPRESSIONS AND LOGIC DIAGRAMS

The previous sections have dealt with simplification of expressions, plotting Veitch diagrams, and extracting the simplest expressions from Veitch diagrams. In order to see the total value of these functions, we will determine their results by the step-by-step application of simplification.

If we have the expression

$$\overline{B + (A + \overline{C})(\overline{A} + C)} + \overline{A + C} + AB\overline{C}$$

we may draw the logic diagram

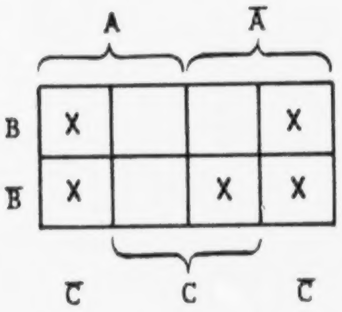

B
A
$\overline{C}$
$\overline{A}$
C
A
C
A
B
$\overline{C}$

We may question whether this diagram is constructed using the fewest gates possible. To determine this, we employ the laws of Boolean algebra to change the given expression to minterm form. We write

$$\overline{B + (A + \bar{C})(\bar{A} + C)} + \overline{A + C} + AB\bar{C}$$

$$= \bar{B}\bar{A}C + \bar{B}A\bar{C} + \bar{A}\bar{C} + AB\bar{C}$$

then plot these terms on a Veitch diagram as



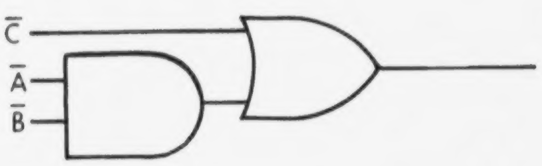
and upon extracting the simplest expression from the plotted squares, we find it to be

$$\bar{A}\,\bar{B} + \bar{C}$$

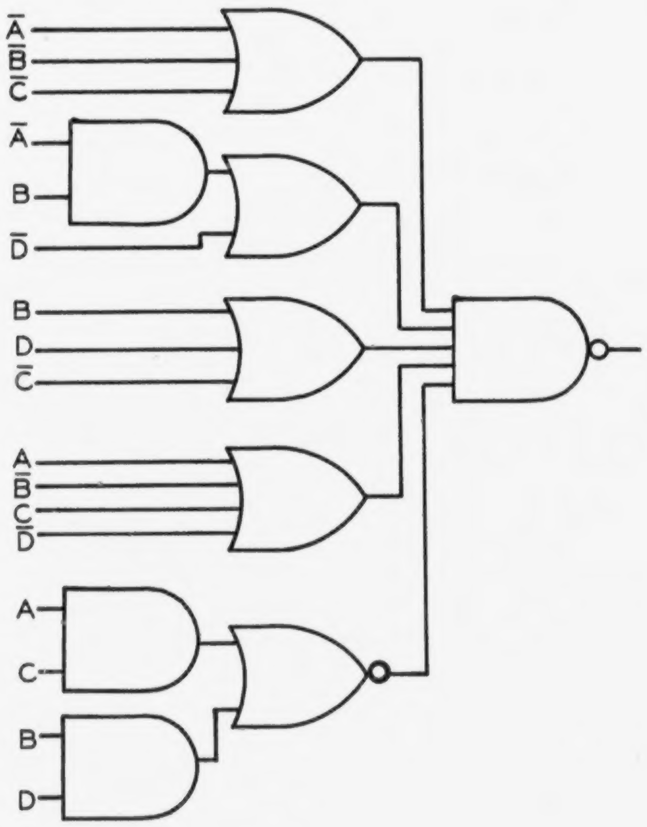
Our next step is to draw the logic diagram for

$$\bar{A}\,\bar{B} + \bar{C}$$

which is



Another example of this technique of simplification is given using the logic diagram



which is identified by the expression

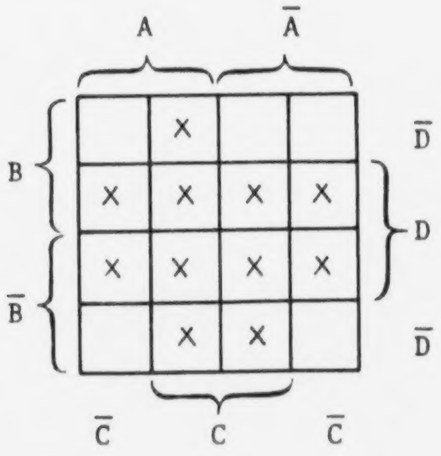
$$\overline{(\bar{A} + \bar{B} + \bar{C})(\bar{A}B + \bar{D})(B + D + \bar{C})}$$

$$\overline{(A + \bar{B} + C + \bar{D})\overline{(AC + BD)}}$$

When written in minterm form, this is

$$ABC + AD + \bar{B}D + \bar{B}\bar{D}C + \bar{A}B\bar{C}D + AC + BD$$

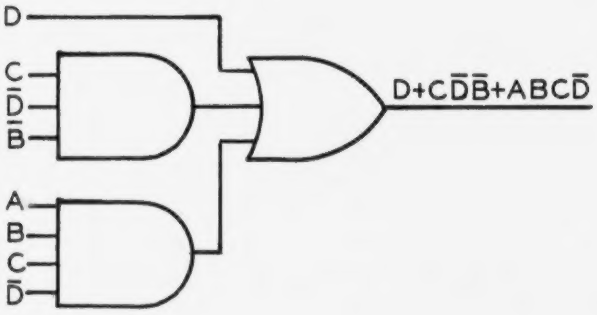
and when plotted, it appears as



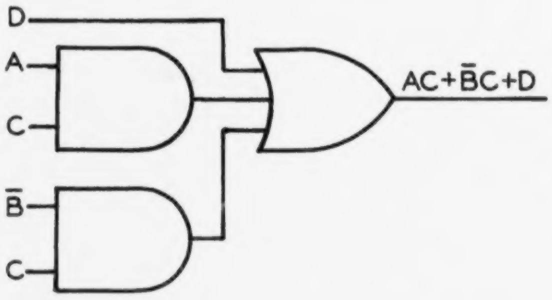
Simplification of these squares results in

$$D + C\bar{D}\bar{B} + ABC\bar{D}$$

which has the logic diagram of


D
C
$\bar{D}$
$\bar{B}$
A
B
C
$\bar{D}$
$D+C\bar{D}\bar{B}+ABC\bar{D}$

This may be further simplified as


D
A
C
$\bar{B}$
C
$AC+\bar{B}C+D$

CHAPTER 9

MATRICES AND DETERMINANTS

While matrix theory, developed in 1858, has many diverse applications, we will direct our discussion toward the objective of solving systems of linear equations.

TERMINOLOGY

We define a matrix as any rectangular array of numbers. We can consider the entries in a table of trigonometric functions as forming a matrix. Also, the entries in a magic square form a matrix. Examples of matrices may be formed from the coefficients and constants of a system of linear equations; that is,

$$2x - 4y = 7$$

$$3x + y = 16$$

can be written

$$\begin{bmatrix} 2 & -4 & 7 \\ 3 & 1 & 16 \end{bmatrix}$$

Notice that we use brackets to enclose the matrix. We could also use double lines; that is,

$$\begin{Vmatrix} 2 & -4 & 7 \\ 3 & 1 & 16 \end{Vmatrix}$$

The numbers used in the matrix are called elements. In the example given we have three columns and two rows. The number of rows and columns are used to determine the dimensions of the matrix. In our example the dimensions of the matrix is 2 x 3. In general, the dimensions of a matrix which has m rows and n columns is called an m x n matrix.

There may occur a matrix with only a row or column in which case it is called either a row or a column matrix. A matrix which has the same number of rows as columns is called a square matrix. Examples of matrices and their dimensions are as follows:

$$\begin{bmatrix} 1 & 7 & 6 \\ 2 & 4 & 8 \end{bmatrix} \quad 2 \times 3$$

$$\begin{bmatrix} 1 & 7 \\ 6 & 2 \\ 3 & 5 \end{bmatrix} \quad 3 \times 2$$

$$\begin{Vmatrix} 2 & 1 \\ 7 & 6 \end{Vmatrix} \quad 2 \times 2 \text{ or square}$$

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad 3 \times 1 \text{ or column}$$

$$\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \quad 1 \times 3 \text{ or row}$$

We will use capital letters, as we did with sets, to describe matrices. We will also include subscripts to give the dimensions; that is,

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 7 & 6 & 5 \end{bmatrix}$$

is the matrix designated by A_{2x3}.

If the situation arises where all of the entries of a matrix are zeros, we call this a zero matrix. The letter we use for a zero matrix is 0. We also include the dimensions; that is, the matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has the designation 0_{3x2}.

We state that two matrices are equal if and only if they have the same dimensions and their corresponding elements are equal. The elements may have a different appearance such as

$$\begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & \frac{1}{1} \\ \frac{6}{3} & 4 \end{bmatrix}$$

but the matrices are equal.

Following are examples of matrices which are equal and matrices which are not equal:

$$\begin{bmatrix} 0 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} \frac{2}{2} & -\frac{6}{2} \\ 4 & 7 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

If we interchange rows and columns of a matrix, we form what is called the transpose of the original matrix. We designate the transpose of matrix B as B^T; that is, if

$$B_{2x3} = \begin{bmatrix} 3 & 4 & 7 \\ 5 & 6 & 9 \end{bmatrix}$$

then

$$B^T = \begin{bmatrix} 3 & 5 \\ 4 & 6 \\ 7 & 9 \end{bmatrix}$$

PROBLEMS: Give the dimensions of the following matrices.

1. $\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 2 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

ANSWERS:

1. 2x3

2. 3x2

3. 2x2 (square)

PROBLEMS: Give the dimensions of the transpose of the previous problem matrices.

ANSWERS:

1. 3x2

2. 2x3

3. 2x2 (square)

Since two matrices are equal if they have the same corresponding elements, we may find an unknown element of one matrix if we know the elements of an equal matrix; that is, if

$$\begin{bmatrix} 0 & 3 & 2 \\ 1 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 \\ x & 7 & 9 \end{bmatrix}$$

then x = 1

PROBLEMS: Find the unknown elements in the following equal matrices.

1. $\begin{bmatrix} x & 2 & 3 \\ 7 & 9 & y \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 9 & 10 \end{bmatrix}$

2. $\begin{bmatrix} 1 & x \\ y & 2 \\ 7 & 11 \end{bmatrix} = \begin{bmatrix} z & 6 \\ 5 & 2 \\ 7 & 11 \end{bmatrix}$

3. $\begin{bmatrix} 0 & x \\ y & z \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 7 \\ 2 & 3 \end{bmatrix}$

ANSWERS:

1. x = 1
   y = 10

2. x = 6
   y = 5
   z = 1

3. x = 1
   y = 4
   z = 7

PROBLEMS: Write the transpose of the following matrices.

1. $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 7 & x \\ 2 & 9 & 11 \end{bmatrix}$

3. $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$

ANSWERS:

1. $[0 \;\; 1 \;\; 2]$

2. $\begin{bmatrix} 3 & 2 \\ 7 & 9 \\ x & 11 \end{bmatrix}$

3. $\begin{bmatrix} x & z \\ y & w \end{bmatrix}$

ADDITION AND SCALAR MULTIPLICATION

We may add only matrices which have the same dimensions. To add matrices we add the corresponding elements and form the sum as a matrix of the same dimension as those added.

EXAMPLE: Add the matrices A and B if

$$A = \begin{bmatrix} 6 & 2 & 7 \\ -1 & 3 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & 6 \end{bmatrix}$$

SOLUTION: Write

$$\begin{bmatrix} 6 & 2 & 7 \\ -1 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 6+2 & 2+1 & 7+3 \\ -1+0 & 3-3 & 0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 3 & 10 \\ -1 & 0 & 6 \end{bmatrix}$$

When we add the zero matrix to any matrix, we find the zero matrix is the identity element for addition; that is,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 2+0 \\ 3+0 & 4+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Also, in addition of numbers we know that a number plus its negative (additive inverse) equals zero; that is,

$$(3) + (-3) = 0$$

This also holds for matrix addition. To form the negative (additive inverse) of a matrix, we write the matrix with the sign of each element changed; that is, if

$$A_{2x3} = \begin{bmatrix} 1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}$$

then its additive inverse is

$$-A_{2x3} = \begin{bmatrix} -1 & -3 & -5 \\ 2 & -6 & -7 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix} + \begin{bmatrix} -1 & -3 & -5 \\ +2 & -6 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 3-3 & 5-5 \\ -2+2 & 6-6 & 7-7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By subtraction of matrices we mean the addition of the additive inverse of the subtrahend; that is,

$$A_{2x2} - B_{2x2}$$

is the same as

$$A_{2x2} + (-B_{2x2})$$

EXAMPLE: Subtract B_{3x2} from A_{3x2} if

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 7 & 3 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 1 \\ 2 & 7 \\ 6 & 3 \end{bmatrix}$$

SOLUTION: Write

$$B = \begin{bmatrix} 0 & 1 \\ 2 & 7 \\ 6 & 3 \end{bmatrix}$$

and

$$-B = \begin{bmatrix} -0 & -1 \\ -2 & -7 \\ -6 & -3 \end{bmatrix}$$

therefore

$$A - B = A + (-B)$$

then

$$\begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 7 & 3 \end{bmatrix} + \begin{bmatrix} -0 & -1 \\ -2 & -7 \\ -6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 0 & 2 - 1 \\ 5 - 2 & 6 - 7 \\ 7 - 6 & 3 - 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 3 & -1 \\ 1 & 0 \end{bmatrix}$$

PROBLEMS: Carry out the indicated operations.

1. $\begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$

2. $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ 6 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 3 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$

ANSWERS:

1. $\begin{bmatrix} 4 & 3 \\ 3 & -3 \end{bmatrix}$

2. $\begin{bmatrix} 10 \\ 8 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 0 & 5 \end{bmatrix}$

4. $\begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$

When solving an equation, in algebra, we isolate the unknown and combine the remainder of the equation; that is, to find the value of x in

$$x + 3 = 7$$

we add the additive inverse of three to each side of the equation to find

$$x + 3 + (-3) = 7 + (-3)$$

and

$$x = 7 + (-3)$$

$$x = 4$$

In dealing with matrices we use the same approach; that is, to solve for the variable matrix in

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 2 & 1 \end{bmatrix}$$

we first add the additive inverse of

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

to each side to find

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ +1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ +1 & 0 \end{bmatrix}$$

then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 3 & 1 \end{bmatrix}$$

There are two types of multiplication when dealing with matrices. The first is multiplication of a matrix by a constant (scalar). The other is the multiplication of one matrix by another matrix.

When multiplying a matrix by a scalar, we write

scalar K times matrix A

where

$$K = 3$$

and

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}$$

is

$$K\begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}$$

Every element of A is multiplied by K such that

$$K\begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} K2 & K3 \\ K1 & K7 \end{bmatrix}$$

and K = 3; therefore

$$\begin{bmatrix} K2 & K3 \\ K1 & K7 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 3 & 21 \end{bmatrix}$$

PROBLEMS: Multiply each matrix by the given scalar.

1. $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $K = 7$
2. $[1,\ 7,\ x]$, $K = 2$
3. $\begin{bmatrix} 0 & 2 & 3 \\ 1 & 3 & 7 \end{bmatrix}$, $K = 6$
4. $6\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$

ANSWERS:

1. $\begin{bmatrix} 21 \\ 7 \\ 0 \end{bmatrix}$
2. $[2\ \ 14\ \ 2x]$
3. $\begin{bmatrix} 0 & 12 & 18 \\ 6 & 18 & 42 \end{bmatrix}$
4. $\begin{bmatrix} 12 & 6 \\ 18 & 6 \end{bmatrix}$

In order to explain the multiplication of one matrix by another matrix we use the example

$$[a\ \ b]\begin{bmatrix} x \\ y \end{bmatrix}$$

and state that the product is

$$ax + by$$

Another example is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

The element aw + by in the product matrix is found by multiplying each element in the first row of the first matrix by the corresponding element in the first column of the second matrix.

The element in the second row and first column of the product matrix is found by multiplying each element in the second row of the first matrix by the corresponding element in the first column of the second matrix.

The following examples should clarify matrix multiplication.

EXAMPLE: Multiply

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

SOLUTION: Write

$$\begin{bmatrix} 1a + 2c & 1b + 2d \\ 3a + 4c & 3b + 4d \end{bmatrix}$$

EXAMPLE: Multiply

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 \\ 0 & 6 \end{bmatrix}$$

SOLUTION: Write

$$\begin{bmatrix} (1 \times 3) + (2 \times 0) & (1 \times 5) + (2 \times 6) \\ (3 \times 3) + (4 \times 0) & (3 \times 5) + (4 \times 6) \end{bmatrix}$$

$$= \begin{bmatrix} 3 + 0 & 5 + 12 \\ 9 + 0 & 15 + 24 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 17 \\ 9 & 39 \end{bmatrix}$$

EXAMPLE: Multiply

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

SOLUTION: Write

$$\begin{bmatrix} (1 \times 3) + (2 \times 4) + (3 \times 0) \end{bmatrix}$$

$$= \begin{bmatrix} 3 + 8 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \end{bmatrix}$$

If two matrices are to be multiplied together, each row in the first matrix must have the same number of elements as each column of the second matrix.

If the left matrix is an n x 3 matrix, the right matrix must be a 3 x m. The product matrix will then be an n x m matrix.

It should be noted that generally matrix multiplication is not commutative. This is shown by the following:

EXAMPLE: Multiply

$$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

SOLUTION: Write

$$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (3 \times 2) + (2 \times 1) & (3 \times 1) + (2 \times 4) \\ (1 \times 2) + (0 \times 1) & (1 \times 1) + (0 \times 4) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 11 \\ 2 & 1 \end{bmatrix}$$

and then write

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (2 \times 3) + (1 \times 1) & (2 \times 2) + (1 \times 0) \\ (1 \times 3) + (4 \times 1) & (1 \times 2) + (4 \times 0) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 4 \\ 7 & 2 \end{bmatrix}$$

which is different from the first product. Since multiplication of matrices is not commutative, we must define multiplication as being either right or left multiplication; that is, xy means left multiplication of y by x, and it also means right multiplication of x by y. Therefore, we find if we are to multiply x by y we have two products to choose from; that is, if

$$x = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and

$$y = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$$

then

$$xy = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 14 \\ 13 & 30 \end{bmatrix}$$

and

$$yx = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 14 \\ 19 & 26 \end{bmatrix}$$

The identity matrices for multiplication are those square matrices which have the elements which form the diagonal from upper left to lower right equal 1 while all other entries are equal to 0; that is, the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the 2x2 identity matrix for multiplication. If we multiply

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we find the product to be

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

PROBLEMS: Multiply

1. $\begin{bmatrix} 1 & 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -2 \\ 1 & -6 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix}$

ANSWERS:

1. $[18]$

2. $\begin{bmatrix} 0 & -1 \\ -16 & -11 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Notice that problem number four infers that when xy equals zero, it is not necessary for either x or y to be zero.

We also make the statement that the distributive laws

$$Ax + Ay = A(x + y)$$

and

$$xA + yA = (x + y)A$$

hold as does the associative law

$$A(BC) = (AB)C$$

DETERMINANT FUNCTION

We may evaluate a square 2x2 matrix and associate the matrix with a real number by adding the product of the elements on one diagonal to the negative of the product of the elements on the other diagonal; that is,

$$\begin{bmatrix} a & B \\ c & D \end{bmatrix}$$

may be associated with aD - Bc. We call this number the determinant of the matrix. NOTE: This procedure applies to second-order determinants only. The determinant function of matrix A is given by $\delta(A)$.

EXAMPLE: If

$$A = \begin{bmatrix} -5 & 1 \\ 3 & 2 \end{bmatrix}$$

find $\delta(A)$

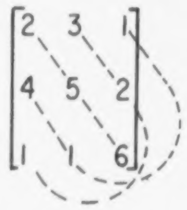
SOLUTION: Write

$$\begin{aligned}\delta(A) &= (-5 \times 2) - (3 \times 1)\\ &= -10 - 3\\ &= -13\end{aligned}$$

We may find the determinant of any square matrix of any dimension. If we have

$$A = \begin{bmatrix} 2 & 3 & 1\\ 4 & 5 & 2\\ 1 & 1 & 6 \end{bmatrix}$$

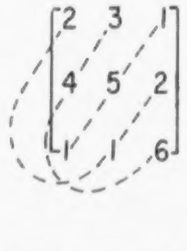
then to find $\delta(A)$ we follow the pattern of



to find the first set of diagonals and write

$$2 \cdot 5 \cdot 6 + 4 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 3$$

We next follow the pattern of



to find the set of diagonals which are to be subtracted from the first set. We write

$$1 \cdot 5 \cdot 1 + 3 \cdot 4 \cdot 6 + 2 \cdot 1 \cdot 2$$

and then

$$\begin{aligned}&(2 \cdot 5 \cdot 6 + 4 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 3)\\ &- (1 \cdot 5 \cdot 1 + 3 \cdot 4 \cdot 6 + 2 \cdot 1 \cdot 2)\\ &= (60 + 4 + 6) - (5 + 72 + 4)\\ &= 70 - 81\\ &= -11\end{aligned}$$

which is the determinant of A. Therefore, if

$$A = \begin{bmatrix} 2 & 3 & 1\\ 4 & 5 & 2\\ 1 & 1 & 6 \end{bmatrix}$$

then

$$\delta(A) = -11$$

NOTE: This pattern applies to third-order determinants only.

In some cases we do not evaluate a matrix to find the determinant but merely write the matrix elements and enclose them by vertical bars; that is, if

$$A = \begin{bmatrix} a & b & c\\ d & e & f\\ g & h & i \end{bmatrix}$$

then

$$\delta(A) = \begin{bmatrix} a & b & c\\ d & e & f\\ g & h & i \end{bmatrix}$$

The order of the determinant is determined by the number of elements in any row or column. In this case the order of the determinant is three.

In the preceding example we may write $\delta(A)$ in the form of

$$(aei + bfg + cdh) - (ceg + bdi + afh)$$

PROBLEMS: Find the determinants of the following matrices.

1. $\begin{bmatrix} 3 & 2\\ -1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 4\\ 5 & 6 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 3 & 2\\ 1 & 1 & 4\\ 1 & 1 & 2 \end{bmatrix}$

ANSWERS:

1. 8
2. -2
3. 4

INVERSE OF A MATRIX

When we multiply two matrices together and find that the product is the multiplicative identity, we say that one of the matrices is the inverse of the other. This is similar to arithmetic in which

$$\frac{1}{a} \cdot \frac{a}{1} = 1$$

and we find $\frac{1}{a}$ is the inverse of a. Notice that this holds as long as "a" does not equal zero. This same requirement is made with matrices; that is, a matrix has an inverse as long as the determinant of the matrix is not equal to zero.

If we multiply the matrices A and B where

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

we find

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

therefore the matrix A is the inverse of matrix B and matrix B is the inverse of matrix A.

Generally, to designate an inverse of a matrix we write, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the inverse of A is

$$A^{-1} = \frac{1}{\delta(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The reason for multiplying by $\frac{1}{\delta(A)}$ is shown by the following example.

EXAMPLE: Find the inverse of matrix A if

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

SOLUTION: Interchange the 1 and 2 and then change the signs of 3 and 4. Write

$$\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}$$

If we now multiply

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}$$

we find the product to be

$$\begin{bmatrix} 2 - 12 & -6 + 6 \\ 4 - 4 & -12 + 2 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix}$$

In order to make this product equal the multiplicative inverse, we multiply by $\frac{1}{-10}$ which gives

$$\frac{1}{-10} \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-10}{-10} & 0 \\ 0 & \frac{-10}{-10} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Also notice that the determinant $\delta(A)$ of

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

is

$$(2) - (12) = -10$$

We may now write the general formula for the inverse of a matrix A as follows. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$A^{-1} = \frac{1}{\delta(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

EXAMPLE: Find the inverse of matrix A if

$$A = \begin{bmatrix} 5 & 6 \\ 4 & 3 \end{bmatrix}$$

SOLUTION: Write

$$A^{-1} = \frac{1}{\delta(A)} \begin{bmatrix} 3 & -6 \\ -4 & 5 \end{bmatrix}$$

and

$$\begin{aligned} \delta(A) &= 15 - 24 \\ &= -9 \end{aligned}$$

Then

$$\begin{aligned} A^{-1} &= \frac{1}{-9} \begin{bmatrix} 3 & -6 \\ -4 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{9} & \frac{2}{3} \\ \frac{4}{9} & -\frac{5}{9} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{4}{9} & -\frac{5}{9} \end{bmatrix} \end{aligned}$$

To verify this we write

$$\begin{aligned} &\begin{bmatrix} 5 & 6 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{4}{9} & -\frac{5}{9} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{5}{3} + \frac{24}{9} & \frac{10}{3} - \frac{30}{9} \\ -\frac{4}{3} + \frac{12}{9} & \frac{8}{3} - \frac{15}{9} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

PROBLEMS: Find the inverse of the following matrices.

1. $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 5 \\ 0 & 1 \end{bmatrix}$

ANSWERS:

1. $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$

3. $\begin{bmatrix} \frac{1}{4} & -\frac{5}{4} \\ 0 & 1 \end{bmatrix}$

DETERMINANTS

We have previously determined that the difference between a matrix and a determinant is that a matrix is an array of numbers and a determinant represents a particular number. When we found the particular number a determinant represented, we called this operation "expanding the determinant"; that is, to expand the determinant

$$\begin{vmatrix} 1 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 6 & 3 \end{vmatrix}$$

we write

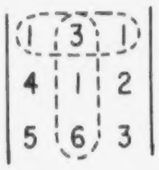
$$\begin{aligned} &(1 \cdot 1 \cdot 3 + 4 \cdot 6 \cdot 1 + 3 \cdot 2 \cdot 5) \\ &- (5 \cdot 1 \cdot 1 + 3 \cdot 4 \cdot 3 + 1 \cdot 6 \cdot 2) \\ &= (3 + 24 + 30) - (5 + 36 + 12) \\ &= 57 - 53 \\ &= 4 \end{aligned}$$

EXPANSION BY MINORS

Another way in which a determinant may be expanded is by expansion by minors. If we have the determinant

$$\begin{vmatrix} 1 & 3 & 1 \\ 4 & 1 & 2 \\ 5 & 6 & 3 \end{vmatrix}$$

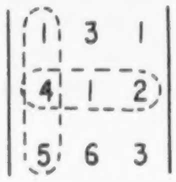
then the minor of the element 3 in the top row is the determinant resulting from the deletion of both row and column that contains the element 3; that is, the minor of 3 is



or

$$\begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix}$$

Also, the minor of the element 4 is



or

$$\begin{vmatrix} 3 & 1 \\ 6 & 3 \end{vmatrix}$$

In order to expand the determinant by minors of the first column we write the minors of 1, 4, and 5 which are:

$$\text{the minor of } 1 = \begin{vmatrix} 1 & 2 \\ 6 & 3 \end{vmatrix}$$

$$\text{the minor of } 4 = \begin{vmatrix} 3 & 1 \\ 6 & 3 \end{vmatrix}$$

$$\text{and the minor of } 5 = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

We now multiply the element by its minor and by the sign of the element location from the pattern

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

which gives

$$1\begin{vmatrix} 1 & 2 \\ 6 & 3 \end{vmatrix}$$

$$-4\begin{vmatrix} 3 & 1 \\ 6 & 3 \end{vmatrix}$$

and

$$5\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

which results in

$$\begin{aligned} 1\begin{vmatrix} 1 & 2 \\ 6 & 3 \end{vmatrix} &= 1\ (3 - 12) \\ &= -9 \end{aligned}$$

and

$$\begin{aligned} -4\begin{vmatrix} 3 & 1 \\ 6 & 3 \end{vmatrix} &= -4\ (9 - 6) \\ &= -12 \end{aligned}$$

and

$$\begin{aligned} 5\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} &= 5\ (6 - 1) \\ &= 25 \end{aligned}$$

which, when added, gives the expansion of the determinant to be

$$- 9 - 12 + 25$$

$$= 4$$

which is the same result as we previously determined.

The expansion of a determinant may be accomplished according to any row or column with the same results. If we expand the determinant

$$\begin{vmatrix} 3 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

about the first row, we have

$$3\begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} -2 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} +1 \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix}$$

$$= 3 + 2 - 2$$

$$= 3$$

If we expand the determinant about the second column, we have

$$-2\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} +4 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} -1 \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 2 + 8 - 7$$

$$= 3$$

It should be noted that a fourth-order determinant has minors which are third-order. The third-order determinants may be defined by second-order determinants. Therefore, through the process of expansion by minors, a high order determinant may finally, through many steps, be expressed by second-order determinants.

PROBLEMS: Expand the following determinants by minors about the row or column indicated. Check your solution by expanding about any other row or column.

1. $\begin{vmatrix} 3 & 1 & 2 \\ 4 & 5 & 6 \\ 0 & 1 & 4 \end{vmatrix}$ about row 1.

2. $\begin{vmatrix} -2 & 1 & -3 \\ 2 & 3 & 1 \\ 1 & -1 & 1 \end{vmatrix}$ about row 3.

3. $\begin{vmatrix} 3 & 5 & 2 \\ 4 & 0 & 1 \\ -1 & 2 & -2 \end{vmatrix}$ about column 2.

ANSWERS:

1. 34

2. 6

3. 45

PROPERTIES

The following properties of determinants may be used to simplify the determinants. These properties apply to any order determinants but will be given with examples using third-order determinants.

(1) The determinant is zero if two rows or two columns are identical.

(2) The sign of the value of a determinant is changed if two rows or two columns of the determinant are interchanged.

(3) The value of a determinant is not changed if all rows and columns are interchanged in order.

(4) The value of a determinant is zero if every element in a row or column is zero.

(5) The value of a determinant is increased by the factor K if any row or column is multiplied by K.

(6) The elements of any row or column may be multiplied by a real number K and these products then added to the elements of another row or column respectively without changing the value of the determinant.

Examples for the listed properties are as follows:

(1)
$$\begin{aligned} \begin{vmatrix} 3 & 3 & 1 \\ 2 & 2 & 6 \\ -1 & -1 & 5 \end{vmatrix} &= (30 - 18 - 2) - (-2 + 30 - 18) \\ &= 30 - 18 - 2 + 2 - 30 + 18 \\ &= 0 \end{aligned}$$

(2)
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ -1 & 2 & 3 \end{vmatrix} \begin{aligned} &= (3 - 10 + 24) - (-3 + 24 + 10) \\ &= 3 - 10 + 24 + 3 - 24 - 10 \\ &= -14 \end{aligned}$$

and

$$\begin{vmatrix} 4 & 1 & 5 \\ 1 & 2 & 3 \\ -1 & 2 & 3 \end{vmatrix} \begin{aligned} &= (24 + 10 - 3) - (-10 + 24 + 3) \\ &= 24 + 10 - 3 + 10 - 24 - 3 \\ &= 14 \end{aligned}$$

(3)
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 3 \\ 2 & 2 & 4 \end{vmatrix} \begin{aligned} &= (4 + 24 + 12) - (6 + 32 + 6) \\ &= 4 + 24 + 12 - 6 - 32 - 6 \\ &= -4 \end{aligned}$$

and

$$\begin{vmatrix} 1 & 4 & 2 \\ 2 & 1 & 2 \\ 3 & 3 & 4 \end{vmatrix} \begin{aligned} &= (4 + 24 + 12) - (6 + 6 + 32) \\ &= 4 + 24 + 12 - 6 - 6 - 32 \\ &= -4 \end{aligned}$$

(4)
$$\begin{vmatrix} 4 & 0 & 6 \\ 2 & 0 & 3 \\ 1 & 0 & 5 \end{vmatrix} \begin{aligned} &= (0 + 0 + 0) - (0 + 0 + 0) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

(5)
$$\begin{vmatrix} 2 & 1 & 1 \\ 4 & 5 & 6 \\ 2 & 1 & 3 \end{vmatrix} \begin{aligned} &= (30 + 4 + 12) - (10 + 12 + 12) \\ &= 30 + 4 + 12 - 10 - 12 - 12 \\ &= 12 \end{aligned}$$

and if K = 2 then

$$\begin{vmatrix} 4 & 1 & 1 \\ 8 & 5 & 6 \\ 4 & 1 & 3 \end{vmatrix} \begin{aligned} &= (60 + 24 + 8) - (20 + 24 + 24) \\ &= 60 + 24 + 8 - 20 - 24 - 24 \\ &= 24 \end{aligned}$$

(6)
$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 3 \end{vmatrix} \begin{aligned} &= (12 + 3 + 1) - (6 + 3 + 2) \\ &= 12 + 3 + 1 - 6 - 3 - 2 \\ &= 5 \end{aligned}$$

and

$$\begin{vmatrix} 2\cdot 2 & 1 & 1 \\ 2\cdot 1 & 2 & 1 \\ 2\cdot 3 & 1 & 3 \end{vmatrix}$$

then

$$\begin{vmatrix} 2 & 1 & 1+4 \\ 1 & 2 & 1+2 \\ 3 & 1 & 3+6 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 9 \end{vmatrix} \begin{aligned} &= (36 + 9 + 5) - (30 + 6 + 9) \\ &= 36 + 9 + 5 - 30 - 6 - 9 \\ &= 5 \end{aligned}$$

EXAMPLE: Evaluate

$$\begin{vmatrix} 1 & 3 & 7 \\ 2 & 9 & 21 \\ -3 & 20 & 16 \end{vmatrix}$$

SOLUTION: It is obvious that if we expand by minors we will encounter large numbers; therefore, we will use some of the properties of determinants.

(1) Multiply the first row by -2 and add product to the second row to find

$$\begin{vmatrix} 1 & 3 & 7 \\ 2 - 2 & 9 - 6 & 21 - 14 \\ -3 & 20 & 16 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 7 \\ 0 & 3 & 7 \\ -3 & 20 & 16 \end{vmatrix}$$

then multiply the first row by 3 and add product to the third row to find

$$\begin{vmatrix} 1 & 3 & 7 \\ 0 & 3 & 7 \\ -3+3 & 20+9 & 16+21 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 7 \\ 0 & 3 & 7 \\ 0 & 29 & 37 \end{vmatrix}$$

Now we expand by minors about the first column; that is,

$$\begin{vmatrix} 1 & 3 & 7 \\ 0 & 3 & 7 \\ 0 & 29 & 37 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 7 \\ 29 & 37 \end{vmatrix} -0 \begin{vmatrix} 3 & 7 \\ 29 & 37 \end{vmatrix} +0 \begin{vmatrix} 3 & 7 \\ 3 & 7 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 7 \\ 29 & 37 \end{vmatrix} = 111 - 203$$

$$= -92$$

EXAMPLE: Evaluate

$$\begin{vmatrix} 1 & 3 & -2 \\ 6 & 7 & 21 \\ 1 & 3 & -2 \end{vmatrix}$$

SOLUTION: The first and third rows are identical; therefore, the determinant value is zero.

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

Second-order determinants may be used to solve systems of two linear equations in two unknowns. If we have two equations such as

$$ax + by = c$$

$$a_1x + b_1y = c_1$$

we may write

$$x = \frac{\begin{vmatrix} c & b \\ c_1 & b_1 \end{vmatrix}}{\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}}$$

and

$$y = \frac{\begin{vmatrix} a & c \\ a_1 & c_1 \end{vmatrix}}{\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}}$$

Note that the denominators are the coefficients of x and y in the same arrangement as given in the problem. Also note that the numerator is formed by replacing the column of coefficients of the desired unknown by the column of constants or the right side of the equations.

EXAMPLE: Solve the system

$$4x + 2y = 5$$

$$3x - 4y = 1$$

SOLUTION: Write

$$x = \frac{\begin{vmatrix} 5 & 2 \\ 1 & -4 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 3 & -4 \end{vmatrix}} = \frac{-20 - 2}{-16 - 6} = \frac{-22}{-22} = 1$$

and

$$y = \frac{\begin{vmatrix} 4 & 5 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 3 & -4 \end{vmatrix}} = \frac{4 - 15}{-16 - 6} = \frac{-11}{-22} = \frac{1}{2}$$

The method of solving the system in this example is called Cramer's rule; that is, when we solve a system of linear equations by the use of determinants, we are using Cramer's rule.

PROBLEMS: Solve for the unknown in the following systems by use of Cramer's rule.

1. $x + 2y = 4$
   $-x + 3y = 1$
2. $3x + 2y = 12$
   $4x + 5y = 2$
3. $x - 2y = -1$
   $2x + 3y = 12$

ANSWERS:

1. $x = 2$
   $y = 1$

2. $x = 8$
   $y = -6$

3. $x = 3$
   $y = 2$

Cramer's rule may be applied to systems of three linear equations in three unknowns. We use the same technique as given in previous examples; that is, if we have

$$ax + by + cz = d$$

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

we may write

$$x = \frac{\begin{vmatrix} d & b & c \\ d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a & d & c \\ a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} a & b & d \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}$$

EXAMPLE: Use Cramer's rule to solve the system

$$2x + 3y - z = 2$$

$$x - 2y + 2z = -10$$

$$3x + y - 2z = 1$$

SOLUTION: Write

$$x = \frac{\begin{vmatrix} 2 & 3 & -1 \\ -10 & -2 & 2 \\ 1 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & 2 \\ 3 & 1 & -2 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} 2 & 2 & -1 \\ 1 & -10 & 2 \\ 3 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & 2 \\ 3 & 1 & -2 \end{vmatrix}}$$

and

$$z = \frac{\begin{vmatrix} 2 & 3 & 2 \\ 1 & -2 & -10 \\ 3 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & 2 \\ 3 & 1 & -2 \end{vmatrix}}$$

Then, solving for the unknowns find that

$$x = -2$$

$$y = 1$$

$$z = -3$$

It should be noted that when determining either the numerator or the denominator in solving systems similar to the previous example, the properties of determinants should be used when possible.

PROBLEMS: Find the solution to the following systems of linear equations.

1. $x + 2y - 3z = -7$
   $3x - y + 2z = 8$
   $2x - y + z = 5$

2. $2x - 3y - 5z = 5$
$x + y - z = 2$
$x - 2y - 3z = 3$

3. $x - 3y - 3z = -2$
$3x - 2y + 2z = -3$
$2x + y - z = 5$

ANSWERS:

1. $x = 1$
$y = -1$
$z = 2$

2. $x = -1$
$y = 1$
$z = -2$

3. $x = 1$
$y = 2$
$z = -1$

INDEX

Absorption, Boolean law, 141, 148
Addition:
  binary, 77
  decimal, 74
  duodecimal, 79
  hexadecimal, 80
  matrix, 164
  octal, 78
  quinary, 75
Algebra:
  binary, 124
  Boolean, 108, 122, 134
  of sets, 117, 120
American Standard Logic Symbols, 130
Analysis of logic circuits by Veitch diagrams, 159
AND operation, 124, 142, 145
Approximation of binomial distribution, 64
Arithmetic:
  mean, 3, 30
  sequences, 1
  series, 4
[illegible]ociative law, 117, 120, 135

Base, number, 71
BCD, 105
Binary:
  addition, 77
  base, 71
  Boolean algebra, 124
  Coded Decimals, 105
  division, 94
  multiplication, 89
  subtraction, 83
  system, 72
Binomial:
  distribution, probability, 52
  expansion, 22, 27
  theorem, 22
Boolean:
  algebra, 108, 122-134
  complements, 147
  expressions, 159
  form of distributive law, 148
  logic, 122
  simplification, 142
Boole, George, 122

Cardinal number, 108
Central tendency measurement (statistics), 30
Choice, principle of, 48
Circuits:
  logic, analysis by Veitch diagrams, 159
  switching, 124
Classes, set theory, 122
Coding, statistical, 32
Combinations, 48
Commutative law, 118, 120, 134
Comparability of sets, 111
Comparison test for series convergence, 17
Complement:
  Boolean, 147
  laws, 119
  set, 116
  subtraction, 85
Complementary law, 137
Computation of statistical mean by coding, 32
Convergence, series, 15-18
Conversions, 96
Cramer's rule, 175
Curve, probability:
  exponential, 57
  normal, 59

Decimal:
  addition, 74
  conversion to BCD, 105
  conversion to nondecimal, 98
  division, 93
  multiplication, 88
  subtraction, 81
  system, 71
Deducing inputs from outputs, 132
DeMorgan's:
  laws, 119
  theorem, 137, 146
Dependent events, 51
Determinant:
  expansion, 171
  function, 168
  properties, 173
Deviation:
  mean, 42
  standard, 42-47, 66
  statistical, 32

Diagrams:
  logic, 159
  Veitch, 150
  Venn, 112, 122
Difference of sets, 115
Disjoint sets, 112
Distribution:
  frequency, statistics, 29
  normal, probability, 57
  Poisson, 61-64
  statistical, 52
Distributive:
  identity, 118
  law, 118, 138
  law, Boolean functions, 148
Divergent series, 15
Division:
  binary, 94
  decimal, 93
  duodecimal, 95
  hexadecimal, 95
  octal, 94
  process, 92
  quinary, 94
Double negation law, 135
Duality, principle of, 119, 120
Dualization, law of, (DeMorgan's theorem), 137
Duodecimal:
  addition, 79
  division, 95
  multiplication, 91
  subtraction, 84
  system, 74

Elements, 122
Equality of sets, 109
Equations, linear, solution of systems of, 175
Excess three code, 106
Exclusive events (probability), 51
Expansion of:
  binomials, 22, 27
  determinants, 172
Exponential curve, 57
Expression, order of, 142

Finite sets, 108
Formula:
  arithmetic mean, 3
  arithmetic series, 5
  geometric series, 10
  infinite geometric series, 13
  summation, arithmetic series, 5
Fractional exponent in a binomial expansion, 25
Frequency distribution, statistics, 29
Function, determinant, 168

General term of a binomial expansion, 24
Geometric:
  mean, 9, 39
  sequence, 7
  series:
    particular term, 7
    summation, 11

Harmonic mean, 40
Hexadecimal:
  addition, 80
  division, 95
  multiplication, 92
  subtraction, 85
  system, 74
Hindu-Arabic number system, 71
Histogram, statistical, 30

Idempotent laws, 117, 120, 135
Identity:
  associative, 117
  commutative and distributive, 118
  idempotent, 117
  laws, 118, 120, 134
Induction, mathematical, 20
Infinite:
  series, 12-15
  sets, 108
Inputs from outputs, deducing, 132
Interpretation of standard deviation, 66
Intersection:
  law of, 137
  relation to AND operation, 145
  sets, 115
Inverse of matrix, 170

Law:
  absorption, Boolean, 141, 148
  associative, 117, 120, 135
  commutative, 118, 120, 134
  complement, 120, 137
  DeMorgan's, 119-120
  distributive, 118, 120, 138
  double negation, 135
  dualization, 137
  idempotent, 117, 120, 135
  identity, 118, 120, 134
  intersection, 137
  union, 137
Laws:
  algebra of sets, 117, 120
  Boolean algebra, 134
Limit of sum, infinite series, 15
Linear equations, solution of systems of, 175

Logic:
    Boolean, 122
    circuits reduced by Veitch diagrams, 160
    diagrams, 159
    symbols, 130

Mathematical induction, 20
Matrix:
    addition, 164
    determinant of, 168
    inverse, 170
    multiplication, 166
    theory, 162
    terminology, 162
Maxterms, 123
Mean:
    arithmetic, 3
    deviation, 33, 41
    geometric, 9, 39
    harmonic, 40
Measure of:
    central tendency, 30
    variability, 41
Median, statistical, 37
Minors, expansion by, 172
Minterm:
    Boolean, 123
    Veitch diagram, 155
Mode, statistical, 38
Multiplication:
    binary, 89
    decimal, 88
    duodecimal, 91
    hexadecimal, 92
    matrix, 166
    octal, 91
    quinary, 89
    scalar, 166
Mutually exclusive events, 51

NAND operation, 128
Negative exponent in a binomial expansion, 25
Normal:
    curve, probability, 58, 65
    distribution, 57
NOR operation, 127
NOT operation, 126
Novenary number system, 71
Nth term of a series, 14
Null:
    class, 122
    set, 109
Number:
    cardinal, 108
    systems, 71

Octal:
    addition, 78
    division, 94
    multiplication, 91
    numbers, 71
    subtraction, 83
    system, 73
Operation:
    AND and OR, 124, 142
    NAND, 128
    NOR, 127
    NOT, 126
    OR, 124
    sets, 114
Order of expression, 142
OR operations, 124, 142
Output used as input, 129

Parentheses in Boolean expressions, 144
Pascal's triangle, 27
Patterns recurring in Veitch diagrams, 157
Permutations, 49
Plotting Veitch diagrams, 151
Poisson distribution, 61
Polygon method of graphing statistical frequency distribution, 30
Postulates, 134
Power set, 111
Principle of:
    choice, 48
    duality, 119
Probability:
    curves, 57-61
    distributions, 52
    fundamentals, 49
Proof of the binomial theorem, 26
Proper subset, 110
Properties of determinants, 173

Quaternary numbers, 71
Quinary:
    addition, 75
    division, 94
    multiplication, 89
    subtraction, 82
    system, 72

Radix, 71
Random sample, 68
Range, statistical, 38
Ratio test for series convergence, 18
Reducing Boolean expressions by Veitch diagrams, 153
Rule:
    Cramer's, 175
    for expanding binomials, 22

Sample frequency distribution (statistics), 32
Sampling, statistical, 68
Scalar multiplication, 166
Scores, standard, 66
Senary numbers, 71
Septenary numbers, 71
Sequences:
    arithmetic, 1
    geometric, 7
Series:
    arithmetic, 4
    convergent, 15
    geometric, 10
    infinite, 12
    n^{th} term, 14
Set:
    comparability, 111
    complement, 116
    difference, 115
    disjoint, 112
    equality, 109
    finite, 108
    infinite, 108
    intersection, 115
    laws, 117, 120
    null, 109
    operations, 114
    power, 111
    union, 114
    universal, 111
Sigma, summation symbol, 32
Simplification, Boolean, 142
Simplifying AND operations, 145
Solution of systems of linear equations, 175
Standard:
    deviation, statistics, 42-47, 66
    logic symbols, 130
    scores, statistics, 66
Statistical:
    coding, 32
    distribution, 52
    frequency, 29
    median, 37
    mode, 38
    range, 38
    sampling, 68
    variability, 41
Subset:
    comparability, 111
    proper, 110
Subtraction:
    binary, 83
    by complements, 85
    decimal, 81
    duodecimal, 84
    hexadecimal, 85
    octal, 83
    quinary, 82
Summation:
    arithmetic series, 5
    geometric series, 11
    symbol, sigma, 32
Switching circuits, 124
Symbols, logic, 130
System, number:
    binary, 72
    decimal, 71
    duodecimal, 74
    hexadecimal, 74
    Hindu-Arabic, 71
    octal, 73
    quinary, 72
Systems of linear equations, solution of, 175

Term:
    geometric series, 7
    particular, series, 14
Terminology, matrix, 162
Ternary, 71
Tests for convergence of series, 17, 18
Theorem:
    binomial, 22
    DeMorgan's, 137
Theory, matrix, 162

Union:
    law of, 137
    relation to OR operations, 145
    sets, 114
Universal:
    class, 122
    set, 111
Use of Veitch diagrams, 152

Variability, statistical, 41
Veitch diagrams, 150, 157
Venn diagrams, 112, 122
Vicenary numbers, 71
Vigesimal numbers, 71

☆ U.S. GOVERNMENT PRINTING OFFICE : 1969 394-048/11